Polynomial Root Distribution and its Impact on Solutions to Thue Equations

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Thesis and Outline

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Outline

- Rational approximations to algebraic numbers (Diophantine approximation).
- The impact of Diophantine approximation on bounding the number of solutions to Thue equations.
- A more general study of polynomial root distribution in the complex plane.

The Density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem (Classical)

Let $\alpha \in \mathbb{R}$ and let $\varepsilon > 0$. Then there exist integers p and q with $q > 0$ so that

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- Pro: this relates easy-to-understand numbers (rational numbers) hard-to-understand numbers (real numbers).
- Con: this theorem is *ineffective*. It gives no indication of how large p or q must be to attain the desired accuracy.

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Benefit: to obtain ε-accuracy, we only need to find q as large as $\frac{1}{\varepsilon}$.

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Benefit: $|\alpha - p/q| < 1/q^2$ is independent of ε.

Distinguishing Between Rationals and Irrationals

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The number of distinct rationals which satisfy $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$ will turn out to be a key feature which indicates whether or not α is rational or irrational.

Correspondence

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Connection to Rationals

We will regularly use the one-to-one correspondence

$$\{\text{primitive pairs } (p, q) \in \mathbb{Z} \times \mathbb{Z}_{>0}\} \to \mathbb{Q}$$
$$(p, q) \mapsto \frac{p}{q}$$

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Corollary (Dirichlet, 1842)

Let $\alpha \in \mathbb{R}$. Then α is irrational if and only if there are infinitely many primitive pairs $(p, q) \in \mathbb{Z}^2$ with $q > 0$ so that

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Key to understanding: the exponent on q in the error term.

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Up Next

Let's look at how that exponent changes according to properties of the number α.

Rational Approximation of Rationals

Fact

Let $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$. Then there are infinitely many primitive pairs $(p, q) \in \mathbb{Z}^2$ with $q > 0$ so that

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If $\varepsilon > 0$, then there are only finitely many primitive pairs $(p, q) \in \mathbb{Z}^2$ with $q > 0$ so that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{1+\varepsilon}}.$$

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Question

Since we have approximation results which distinguish between rational numbers and irrational numbers, can we find an approximation result which distinguishes between algebraic and transcendental numbers?

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Theorem (Roth, 1955)

Suppose that $\alpha \in \mathbb{R}$ is algebraic and $\varepsilon > 0$. Then there are only finitely many primitive pairs $(p, q) \in \mathbb{Z}^2$ with $q > 0$ so that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}}.$$

Approximation Summary

Facts

- Real numbers α are classified by the "largest" exponent μ which allows infinitely many rational solutions to

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Problem

For $\delta > \mu$, how do we find the finitely many rational solutions to

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^\delta}?$$

Intro to Thue Equations

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Definition

Let $F(x, y)$ be an integral binary form which is irreducible over $\mathbb{Q}$ and has degree at least 3. Let h be an integer. Then the equation

$$F(x, y) = h$$

is known as a *Thue equation* and the inequality

$$|F(x, y)| \leqslant h$$

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Therefore, for some i, $\left|\frac{p}{q} - \alpha_i\right|$ is small.

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Therefore, for some i, $\left|\frac{p}{q} - \alpha_i\right|$ is small. Now we can use Diophantine approximation results to show that the number of solutions is finite.

Follow-up Questions

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- What are the solutions to $F(x, y) = h$?
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- On which features of $F(x, y)$ and h do the number of solutions depend?

Manageable Follow-up Questions

Theorem (Thue, 1909)

There are finitely many integer solutions to any Thue inequality, $|F(x,y)| \leqslant h$.

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We'll answer the last question first.

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- Example: $F(x, y) = x^6 - 2x^4y^2 + 10x^2y^4 + 10y^6$

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  - Set $H = \max_i |a_i|$ to be the height of F.
- Example: $F(x, y) = x^6 - 2x^4y^2 + 10x^2y^4 + 10y^6$
  - $n = 6$
  - $s = 3$
  - $H = 10$
- Let $h \in \mathbb{Z}_{>0}$

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Note

This is where polynomial root *distribution* impacts solutions to Thue equations.

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Lemma (Descartes, 1637)

If $g(x) \in \mathbb{R}[x]$ has $s + 1$ nonzero summands, then $g(x)$ has no more than $2s + 1$ real roots.

Parameter Summary

Goal

Bound the number of (integer) solutions to $|F(x, y)| \leqslant h$ in terms of n (the degree of F), s (the number of nonzero summands of F), and h.

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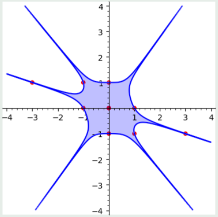
Notation

Let $N(F,h)$ denote the number of integer solutions to the Thue inequality $|F(x,y)| \leqslant h$.

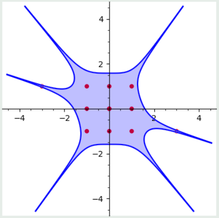
Geometric View of $|F(x,y)| \leqslant h$

A Picture

$|F(x,y)| \leqslant h$ corresponds to a region of the xy-plane:



$|x^5 + 3x^4y - y^5| \leqslant 1$



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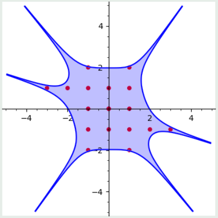

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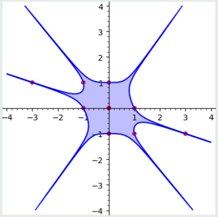

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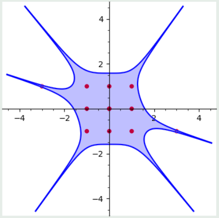

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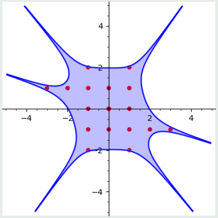

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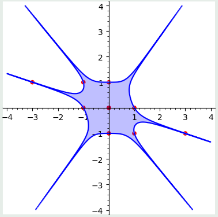

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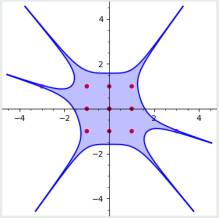

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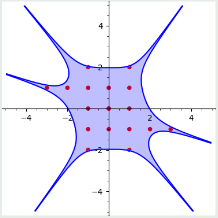

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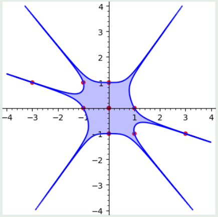
h	1	10	30
$N(F,h)$	9	11	17

Exploring Dependence on h

A Picture



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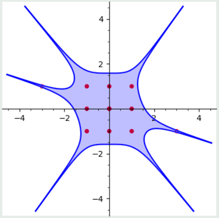

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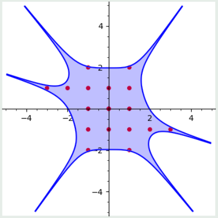

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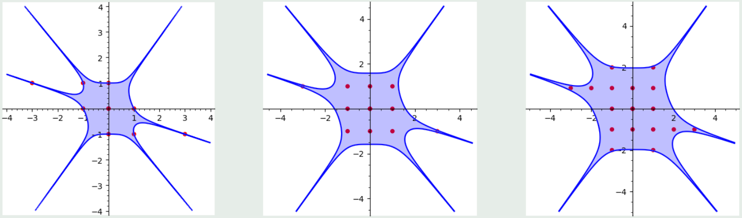

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$$N(F,h) = \text{number of lattice points ``inside'' } |F(x,y)| \leqslant h$$

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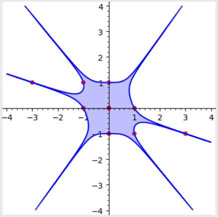

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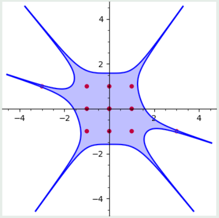

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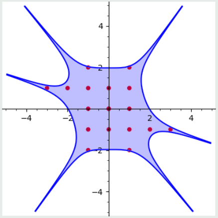

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$$N(F,h) = \text{number of lattice points "inside" } |F(x,y)| \leqslant h$$
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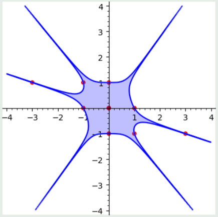

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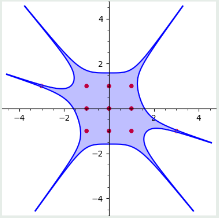

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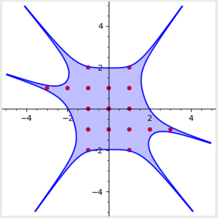

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$$\begin{aligned} N(F,h) &= \text{number of lattice points "inside" } |F(x,y)| \leqslant h \\ &\approx \operatorname{vol}\{(x,y) \in \mathbb{R}^2 : |F(x,y)| \leqslant h\} \\ &= h^{2/n} \operatorname{vol}\{(u,v) \in \mathbb{R}^2 : |F(u,v)| \leqslant 1\} \end{aligned}$$

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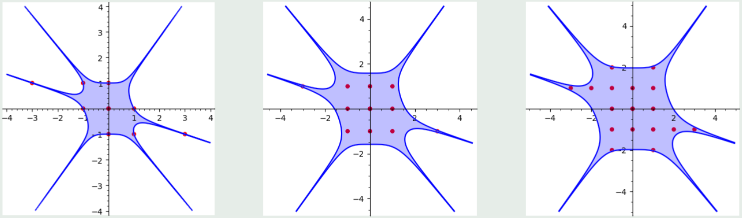

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Exploring dependence on h

Theorem (Mahler, 1934)

Let

$$V(F,1) := \mathrm{vol}\{(x,y) \in \mathbb{R}^2 : |F(x,y)| \leqslant 1\}.$$

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Moral

The factor of $h^{2/n}$ is necessary and sufficient and we expect

$$N(F,h) \approx h^{2/n} \cdot N(F,1).$$

Exploring $N(F, 1)$

Previous Facts

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Based on some approximation theorems, we expect the number of rational approximations per root to be absolutely bounded.

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Conclusion

We expect there to be no more than a constant times s solutions to $|F(x, y)| \leqslant 1$.

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Meaning

The symbol $\ll$ means "(is) no more than a constant times."

Conclusion (rephrased)

We expect there to be $\ll s$ solutions to $|F(x, y)| \leqslant 1$.

A Conjecture and Theorem of Mueller and Schmidt

The Pieces

Recall that we expect:

$$N(F,h) \approx h^{2/n} \cdot N(F,1)$$
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Picking Values for s and h

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$ax^n - by^n = 1$ *has at most one solution in positive integers* x *and* y*.*

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Theorem (Thomas, 2000)

For $n \geqslant 39$ and $F(x, y) = ax^n + bx^ky^{n-k} + cy^n$, the number of solutions to $|F(x, y)| = 1$ is less than or equal to 48.

Types of Solutions

Separating Solutions

- Begin by choosing some (explicit) constants $0 < Y_S < Y_L$ which depend on F.

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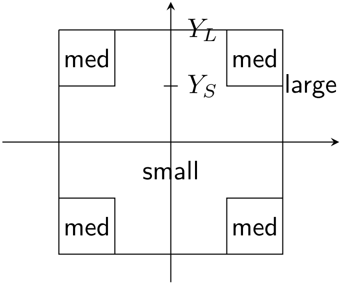

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- Medium solutions to $|F(x, y)| \leqslant h$ produce good approximations to some small set of algebraic numbers (Mueller and Schmidt, 1987).
- Fix one of those algebraic numbers, α, and note that the good rational approximations of α must be close to each other.
- Manipulate inequalities to find that the denominators of the rational functions must be exponentially far apart.

Counting Medium Solutions

Our improvement involves efficiently bounding t in the situation where both

$$Y_S \leqslant q_0 < q_1 < \cdots < q_t \leqslant Y_L$$

and the inductive relation

$$q_{i+1} > \frac{q_i^{\frac{n}{s}-1}}{K}$$

hold.

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hold.

Lemma (K., 2021)

If $n \geqslant 3s$ and there are $t+1$ medium solutions associated to α, then

$$t \leqslant \frac{\log\left[\frac{\log Y_L K^{-1/(\frac{n}{s}-2)}}{\log Y_S K^{-1/(\frac{n}{s}-2)}}\right]}{\log\left(\frac{n}{s}-1\right)}.$$

Moreover, this bound is as sharp as possible, given existing approximation results.

Counting Solutions

Theorem (K., 2021)

The number of primitive medium solutions to $|F(x, y)| \leqslant h$ when $n \geqslant 3s$ is

$$\ll s\left(1 + \log\left(s + \frac{\log h}{\max(1, \log H)}\right)\right)$$

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Bounds for Small and Large Solutions

- The number of large primitive solutions is $\ll s$ (Mueller and Schmidt, 1987).
- The number of small primitive solutions is $\ll se^{\Phi}h^{2/n}$ when $n > 4se^{2\Phi}$ (Saradha and Sharma, 2017).

Note: Φ measures the "sparsity" of F and satisfies $\log^3 s \leqslant e^{\Phi} \ll s$.

Asymptotic Bounds

As a consequence:

Theorem (K., 2023)

When $n > 4se^{2\Phi}$, the number of primitive solutions to $|F(x,y)| \leqslant h$ is

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Recall that $\log^3 s \leqslant e^{\Phi} \ll s$.

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Recall that $\log^3 s \leqslant e^{\Phi} \ll s$.

Compare to:

Theorem (Mueller and Schmidt, 1987)

The number of integer pair solutions to $|F(x,y)| \leqslant h$ is

$$\ll s^2 h^{2/n}(1 + \log h^{1/n}).$$

Explicit Bounds for Trinomials

Theorem (Thomas, 2000)

If $F(x,y) = ax^n + bx^k y^{n-k} + cy^n$*, there are no more than* $C_1(n)$ *solutions to* $|F(x,y)| = 1$ *where* $C_1(n)$ *is defined by*

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Question

Is this a good bound?

Trinomial Computations

H	1	2	3	4	5	6	7	8	9	10	$\cdots$	16
$n = 6$	8	6	8	8	6	6	6	6	8	6	-	12
$n = 7$	8	6	8	8	6	6	6	6	8	6	-	8
$n = 8$	8	6	8	8	6	6	6	6	8	6	-	12
$n = 9$	8	6	8	8	6	6	6	6	8	6	-	8
$n = 10$	8	6	8	8	6	6	6	6	8	-	-	-
$n = 11$	8	6	8	8	6	6	6	6	8	-	-	-
$n = 12$	8	6	8	8	6	6	6	-	-	-	-	-
$n = 13$	8	6	8	8	6	6	-	-	-	-	-	-
$n = 14$	8	6	8	8	6	6	-	-	-	-	-	-
$n = 15$	8	6	8	8	6	-	-	-	-	-	-	-
$n = 16$	8	6	8	8	6	-	-	-	-	-	-	-
$n = 17$	8	6	8	8	-	-	-	-	-	-	-	-

Maximum number of solutions to $|F(x, y)| = 1$ for any trinomial of height H and degree n

Thue Equation Summary

New Results

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Questions

Any question on Thue equations?

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$$f(x) = \sum_{i=0}^{n} b_i x^i$$

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$\lvert \Delta_f \rvert$	$\lvert b_n \rvert^{2n-2} \prod\limits_{1 \leqslant i < j \leqslant n} \lvert \alpha_i - \alpha_j \rvert^2$	discriminant

Useful Fact and Motivating Question

Fact

$$\binom{n}{\lfloor n/2 \rfloor}^{-1} H(f) \leqslant M(f) \leqslant \sqrt{n+1} \cdot H(f)$$

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Question

How are Mahler measure and separation related? Can we find bounds on separation in terms of Mahler measure?

Mahler Measure and Separation

Theorem (Mahler, 1964)

For all monic polynomials $f(x) \in \mathbb{C}[x]$ of degree $n \geqslant 2$,

$$\operatorname{sep}(f) > \frac{\sqrt{3}|\Delta_f|^{1/2}}{n^{(n+2)/2} M(f)^{n-1}}.$$

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Corollary (Mahler, 1964)

For separable monic $f(x) \in \mathbb{Z}[x]$ *of degree* $n \geqslant 2$,

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Others (Bugeaud, Dujella, Pejković) have results when $f(x)$ is assumed to be (ir)reducible.

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Inequality manipulation yields

$$\operatorname{sep}(f) < n^{\frac{1}{n-3}} M(f)^{\frac{2}{n-\frac{1}{2}}} \approx M(f)^{2/n}.$$

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Question

Can we get a better exponent in the relation

$$\operatorname{sep}(f) \ll M(f)^{2/(n-1/2)}?$$

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Algorithm

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"Random" Polynomials

A "random" polynomial is a polynomial whose roots are sampled according to a uniform distribution in an appropriate region of $\mathbb{C}$.

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- We can illustrate the difference between these cases as follows.

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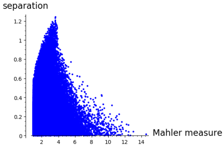
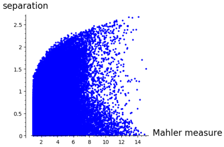


Signature $(4, 0)$



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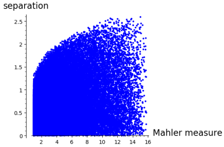

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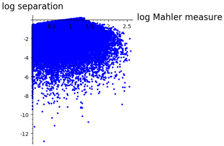

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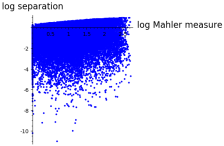

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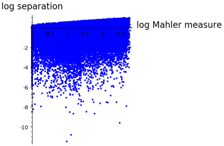

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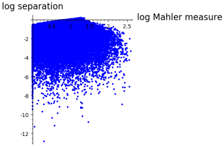

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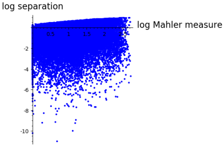

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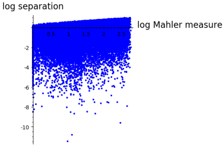
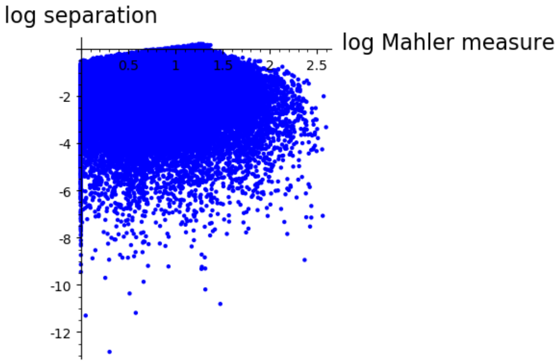


Signature $(0, 2)$

The logarithmic separation appears to be bounded above by a linear function of the logarithmic Mahler measure.

Discerning the Upper Bound: Signature $(4, 0)$



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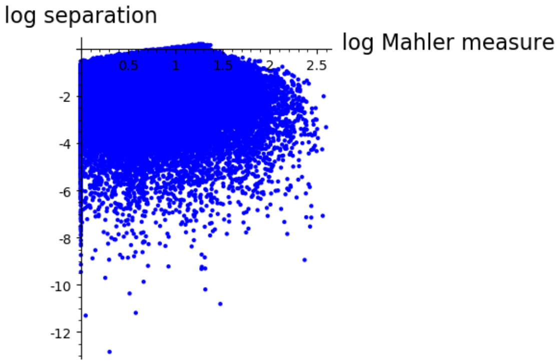

The upper bound here appears to be something like $\log \operatorname{sep}(f) \leqslant \frac{1}{3} \log M(f) - \frac{1}{2}$, i.e.

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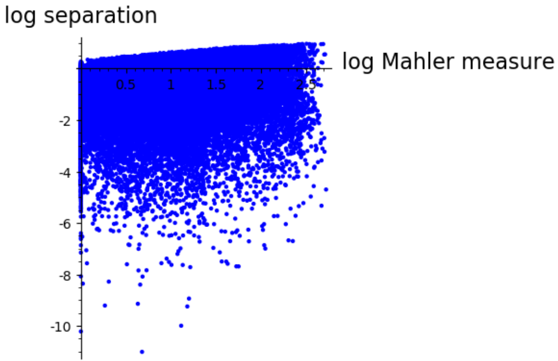

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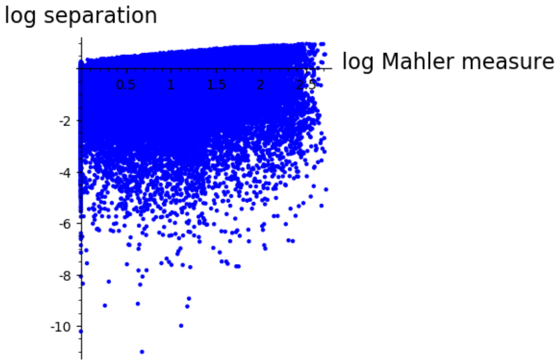

The upper bound in this case appears to be something like $\log \operatorname{sep}(f) \leqslant \frac{1}{3} \log M(f) + \frac{1}{4}$, i.e.

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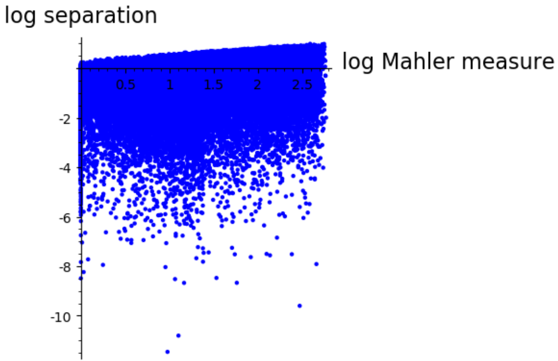
Discerning the Upper Bounds: Signature $(0, 2)$



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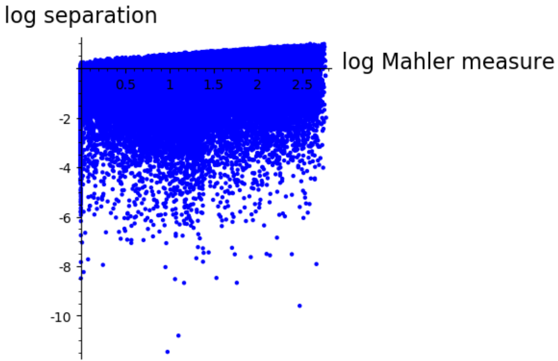

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Here is a plot of the logarithmic separations and Mahler measures of 2,000,000 degree 10 monic polynomials with 5 pairs of complex conjugate roots:

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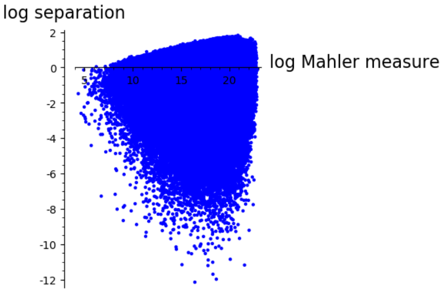

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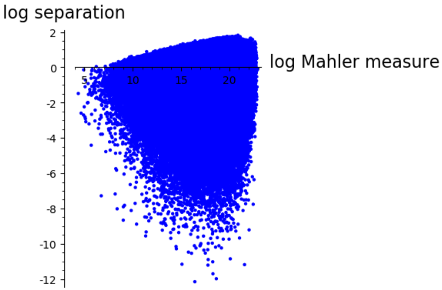

Here, we get something like $\log \operatorname{sep}(f) \leqslant \frac{1}{10}\log M(f) - \frac{1}{2}$, i.e.

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A Conjecture

Conjecture (K., 2023)

Suppose $f(x) \in \mathbb{R}[x]$ is monic of degree n. If $f(x)$ has any real roots, then

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If $f(x)$ has two real roots, then

$$\operatorname{sep}(f) \leqslant \left(\frac{M(f)}{|a|}\right) + 1.$$

Moreover, these bounds are sharp.

The Totally Real Case

The totally real case is also manageable due to the restricted geometry of the root locations:

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Proposition (K., 2023)

Let $f(x) \in \mathbb{R}[x]$ be separable and monic of degree $n \geqslant 3$ and suppose that all n of the roots of f are real. Then

$$\operatorname{sep}(f) \leqslant \frac{8.2}{n^{\frac{n}{n-1}}} \cdot M(f)^{1/(n-1)}.$$

The Remaining Cubic Case

Proposition (K., 2023)

Let $f(x) \in \mathbb{R}[x]$ have degree 3 and leading coefficient a. If $f(x)$ has exactly one real root, then

$$\operatorname{sep}(f) < \sqrt{3}\left(\frac{M(f)}{|a|}\right)^{1/2}.$$

The Quartic with Signature $(0, 2)$ Case

Proposition (K., 2023)

Suppose that $f(x) \in \mathbb{R}[x]$ has degree 4, leading coefficient a and no real roots. Then

$$\operatorname{sep}(f) \leqslant \sqrt{2}\left(\frac{M(f)}{|a|}\right)^{1/4}.$$

Moreover, this bound is sharp.

Summary

Conjecture (K., 2023)

Suppose $f(x) \in \mathbb{R}[x]$ is monic of degree n. If $f(x)$ has any real roots, then $\operatorname{sep}(f) \ll_n M(f)^{1/(n-1)}$. If $f(x)$ has only nonreal roots, then $\operatorname{sep}(f) \ll_n M(f)^{1/n}$.

Theorem (K., 2023)

This conjecture holds for $f(x) \in \mathbb{R}[x]$ which meet any of the following conditions:

1. $\deg(f) = 2$.
2. $\deg(f) = 3$.
3. $\deg(f) = 4$ *and $f(x)$ has no real roots.*
4. *Every root of $f(x)$ is real.*

Thank you!

Questions?