Exponential Relations Among Algebraic Integer Conjugates

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Joint work with Seda Albayrak, Samprit Ghosh, and Khoa Nguyen.

Land Acknowledgment

The University of Calgary, located in the heart of Southern Alberta, both acknowledges and pays tribute to the traditional territories of the peoples of Treaty 7, which include the Blackfoot Confederacy (comprised of the Siksika, the Piikani, and the Kainai First Nations), the Tsuut'ina First Nation, and the Stoney Nakoda (including Chiniki, Bearspaw, and Goodstoney First Nations). The City of Calgary is also home to the Métis Nation of Alberta Districts 5 and 6.

Motivating Result

Theorem (Buguead and Nguyen, 2023)

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Let ξ be an irrational, algebraic number of degree $d \geqslant 3$. Let $\varepsilon > 0$. Let $(u_n)_{n \geqslant 1}$ be a non-degenerate linear recurrence sequence of rational integers which is not a polynomial sequence. Then there are only finitely many u_n for which there exists a $v_n \in \mathbb{Z}$ so that

$$\left|\xi - \frac{v_n}{u_n}\right| < \frac{1}{|u_n|^{1+\frac{1}{d-1}+\varepsilon}}.$$

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Interpretation

Certain sequences cannot serve as denominators for good rational approximations of ξ.

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$$\left| \xi - \frac{v_n}{u_n} \right| < \frac{1}{|u_n|^{1 + \frac{1}{d-1} + \varepsilon}}.$$

Question

Can the exponent of $\frac{1}{d-1}$ be improved (decreased) at all in order to achieve the same result?

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Question

Can we replace $d-1$ by anything else and still have the fact be true?

Other Exponents

Question

Let $d \geqslant 2$ be an integer. For which values of $c \geqslant 0$ is it true that for every monic, irreducible $f(x) \in \mathbb{Z}[x]$ of degree d with roots $\alpha_0, \ldots, \alpha_{d-1} \in \mathbb{C}$ in descending order,

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If the above property holds, then $c \leqslant d - 1$.

- Why?
- Let's look at the family of polynomials $f_{d,h}(x) = x^d - hx^{d-1} - 1$.

A Useful Example

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- $f_{d,h}$ has one "large" root: $|\alpha_0| \asymp |h|$.
- $f_{d,h}$ has $d-1$ "small" roots:

$$|\alpha_1|, \ldots, |\alpha_{d-1}| \asymp |h|^{-1/(d-1)}.$$

$d-1$ Is The Best Possible Exponent

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Summary

Recap

A real number $c \geqslant 0$ has the property that for every irreducible, monic $f(x) \in \mathbb{Z}[x]$,

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Corollary

The exponent in our motivating theorem is optimal.

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Follow-Up Questions

- If this is the "one-dimesional problem," what do the higher-dimensional problems look like?
- For $c \in [0, d-1]$, can we guarantee that $|\alpha_0||\alpha_1|^c > 1$? If so, by how much?

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Question

What is the shape of $E_{k,d}$?

First Perspective

Theorem (Albayrak, Ghosh, K., Nguyen)

$E_{k,d}$ is the set of all points in $\mathbb{R}^k$ which satisfy the following:

$$x_i \geqslant 0 \qquad \text{for } 1 \leqslant i \leqslant k$$

$$-\frac{d-i}{i}\sum_{j=1}^{i-1} x_j + \sum_{j=i}^{k} x_j \leqslant \frac{d-i}{i} \qquad \text{for } 1 \leqslant i \leqslant k$$

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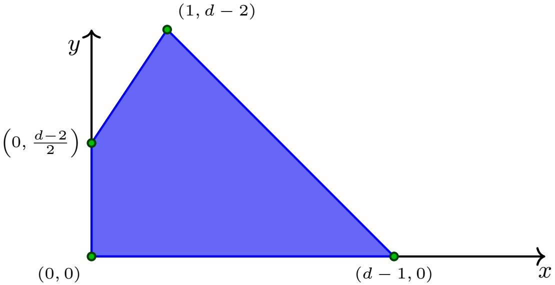
$$-\frac{d-2}{2}x + y \leqslant \frac{d-2}{2} \qquad (i = 2).$$

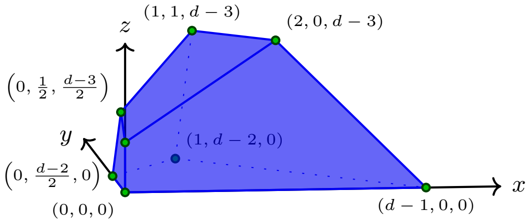
A Picture

A picture of $E_{2,d}$ created in SageMath:



Another Picture

An image of $E_{3,d}$ created in SageMath:



Sources of the Inequalities

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Where do the inequalities of the form

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Answer

- The ith inequality comes from the family of polynomials
$$x^d - hx^{d-i} - 1$$
for $h \in \mathbb{Z}$.

Sources of the Inequalities

Question

Where do the inequalities of the form

$$-\frac{d-i}{i}\sum_{j=1}^{i-1} x_j + \sum_{j=i}^{k} x_j \leqslant \frac{d-i}{i} \qquad \text{for } 1 \leqslant i \leqslant k$$

come from?

Answer

- The ith inequality comes from the family of polynomials
$$x^d - hx^{d-i} - 1$$
for $h \in \mathbb{Z}$.
- For large $|h|$, these polynomials have i roots of size $\approx |h|^{1/i}$ and $d-i$ roots of size $\approx |h|^{-1/(d-i)}$.

Equality and Inequality in $E_{1,d}$

Question

For $c \in [0, d-1]$, is it possible that

$$|\alpha_0||\alpha_1|^c = 1?$$

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Reduction

If $f(x)$ is not cyclotomic, then $|\alpha_0||\alpha_1|^c = 1$ only if $c = d - 1$.

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Nontrivial Answers

- $f(x) = x^2 - x - 1$ has

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- $f(x) = x^3 + x^2 - x + 1$ is such a polynomial.
- If $\deg(f) > 3$, then
$$|\alpha_0||\alpha_1|^{d-1} > 1.$$

Equality and Inequality in General

Theorem (Albayrak, Ghosh, K., Nguyen)

If $d > 3k + 1$ and $(c_1, \ldots, c_k) \in E_{k,d}$, then any monic, irreducible, noncyclotomic $f(x) \in \mathbb{Z}[x]$ with roots $\alpha_0, \ldots, \alpha_{d-1}$ in descending order has

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Note

The lower bound on d is suboptimal for $k = 1$ and $k = 2$.

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$$|\alpha_0||\alpha_1|^{c_1} \ldots |\alpha_k|^{c_k} > 1.$$

Future Work

If $d > 3k + 1$, can we get a lower bound on

$$|\alpha_0||\alpha_1|^{c_1} \ldots |\alpha_k|^{c_k} - 1?$$

Thank you!

Questions?