1. *Prove that no integer in the sequence*

$$11, 111, 1111, 11111, \ldots$$

is a perfect square.

Set $a_n = \sum_{k=0}^{n} 10^k$ and observe that we are trying to show that a_n is not a perfect square when $n \geqslant 1$. Suppose, by contradiction, that there exists $r \in \mathbb{Z}$ so that $a_n = r^2$.

If r is even (say $r = 2s$ for some $s \in \mathbb{Z}$), then $r^2 = 4s^2$ is a multiple of 4. $a_n = 1 + 2\sum_{k=1}^{n} 5^k \cdot 2^{k-1}$, so a_n is odd. Hence, a_n cannot be equal to r^2 when r is even.

Then r must be odd. Problem 3 on the week 2 group work indicates then that r^2 must have the form $4j + 1$ for some integer j. Note that when $n \geqslant 2$, $a_n = 3 + 4\left(2 + \sum_{k=2}^{n} 5^k \cdot 2^{k-2}\right)$, so a_n has the form $3 + 4\ell$ for some $\ell \in \mathbb{Z}$. We cannot have $3 + 4\ell = 1 + 4j$ when $\ell, j \in \mathbb{Z}$ (because if we did, then $2 = 4(j - \ell)$, which is impossible since $4 \nmid 2$), so we cannot have $r^2 = a_n$.

Hence, we have shown that we cannot have $r \in \mathbb{Z}$ with $r^2 = a_n$ when $n \geqslant 2$.

2. *A* <u>*square-free*</u> *integer is an integer that is not divisible by any perfect squares other than 1. Prove that every integer can be written as the product of a square and a square-free integer.*

Observe first that since 0 is a perfect square and 1 is square-free, $0 = 0 \cdot 1$ is the product of a perfect square and a square-free integer. Likewise, since 1 is both a perfect square and a square-free integer, $1 = 1 \cdot 1$ is the product of a perfect square and a square-free integer. Finally, -1 is square-free and hence, $(-1) = 1 \cdot (-1)$ can be written as the product of a square and a square-free integer.

Now fix $n > 1$. By the fundamental theorem of arithmetic, there exist distinct primes $p_1, \dots, p_k$ and integers $e_1, \dots, e_k \geqslant 1$ with $p_1^{e_1} \cdots p_k^{e_k} = n$. For $1 \leqslant i \leqslant k$, define b_i to be 0 if e_i is even and define b_i to be 1 if e_i is odd. Then for each i, $e_i - b_i$ is even. Hence, we can rewrite

$$n = (p_1^{b_1} \cdots p_k^{b_k}) \cdot \left(p_1^{\frac{e_1 - b_1}{2}} \cdots p_k^{\frac{e_k - b_k}{2}} \right)^2$$

$\left(p_1^{\frac{e_1 - b_1}{2}} \cdots p_k^{\frac{e_k - b_k}{2}} \right)^2$ is a perfect square, so it only remains to show that $p_1^{b_1} \cdots p_k^{b_k}$ is square-free. If $\ell \in \mathbb{Z}$ and $\ell^2 \mid (p_1^{b_1} \cdots p_k^{b_k})$, then any prime $p \mid \ell$ must have $p = p_j$ for some $1 \leqslant j \leqslant k$, giving $p_j^2 \mid p_1^{b_1} \cdots p_k^{b_k}$. But this is impossible since each $b_i \leqslant 1$ and the primes $p_1, \dots, p_k$ are distinct and hence, ℓ cannot have any prime factors. So $\ell = \pm 1$ implying that $\ell^2 = 1$. Hence, 1 is the only square dividing $p_1^{b_1} \cdots p_k^{b_k}$ and so $p_1^{b_1} \cdots p_k^{b_k}$ is square-free.

This concludes the proof that any nonnegative integer can be written as the product of a perfect square and a square-free integer. Now suppose that $n < -1$ and note that $|n|$ is strictly larger than 1 and hence, can be written as a product of a perfect square and a square-free integer. Write $|n| = sf$ where s is a perfect square and f is square-free. Then $n = s(-f)$. s is still a perfect square and $-f$ is square-free because f is square-free (after all, any square which divides $-f$ will also divide f).

We conclude that for every integer n, n can be written as the product of a perfect square and a square-free integer.

3. *Show that* $\sqrt[3]{5}$ *is irrational.*

Suppose, by contradiction, that $\sqrt[3]{5}$ is rational. Then there exist integers $a, b \in \mathbb{Z}$ with $b \neq 0$ and $\sqrt[3]{5} = \frac{a}{b}$. Cubing both sides of the equation gives $5 = \frac{a^3}{b^3}$ and hence, $5b^3 = a^3$. By the fundamental theorem of arithmetic, a factors uniquely into primes, implying that the factorization of a^3 has every prime raised to a power which is a multiple of 3. In particular, the power of 5 which divides a^3 must be a multiple of 3. By symmetry, the power of 5 which divides b^3 is also a multiple of 3 and hence, the power of 5 which divides $5b^3$ cannot be a multiple of 3. This contradicts the finding that $5b^3 = a^3$, so it must be the case that $\sqrt[3]{5}$ is irrational.