1. *Let F_n denote the nth Fibonacci number. Show that $F_n = 5F_{n-4} + 3F_{n-5}$ whenever $n \geqslant 5$. Use this result to show that F_n is divisible by 5 whenever n is divisible by 5.*

   Suppose that $n \geqslant 5$. Then we can write

$$\begin{aligned}
F_n &= F_{n-1} + F_{n-2} \\
&= (F_{n-2} + F_{n-3}) + (F_{n-3} + F_{n-4}) \\
&= F_{n-2} + 2F_{n-3} + F_{n-4} \\
&= (F_{n-3} + F_{n-4}) + 2(F_{n-4} + F_{n-5}) + F_{n-4} \\
&= F_{n-3} + 4F_{n-4} + 2F_{n-5} \\
&= (F_{n-4} + F_{n-5}) + 4F_{n-4} + 2F_{n-5} \\
&= 5F_{n-4} + 3F_{n-5}
\end{aligned}$$

   Now suppose that $5 \mid n$, so that we can write $n = 5k$. We prove that $5 \mid F_n$ by induction on k.

   If $k = 0$, then $F_0 = 0$ and since $5 \mid 0$, we have $5 \mid F_0$.

   Now suppose that $5 \mid F_{5k}$ and we aim to show that $5 \mid F_{5(k+1)}$. Since $5(k+1) \geqslant 5$, we can write $F_{5(k+1)} = 5F_{5k+1} + 3F_{5k}$. By the induction hypothesis, F_{5k} is a multiple of 5. Additionally, $5F_{5k+1}$ is a multiple of 5 and hence, $5F_{5k+1} + 3F_{5k}$ is also a multiple of 5. Therefore, $5 \mid F_{5(k+1)}$.

   By induction, $5 \mid F_n$ whenever n is a multiple of 5.

2. We say that a is relatively prime to b if $(a, b) = 1$.

   (a) *Find all positive integers less than 10 that are relatively prime to 10.*

   Note that $(a, b) = 1$ if and only if a and b have different prime factors. So $(10, a) = 1$ if and only if a is neither a multiple of 2, nor a multiple of 5. Hence, the positive integers less than 10 that are relatively prime to 10 are $1, 3, 7$, and 9.

   (b) *Find all positive integers less than 11 that are relatively prime to 11.*

   Since any positive integer less than 11 can't share a prime factor with 11, every positive integer less than 11 is relatively prime to 11.

3. *Are there integers $a, b,$ and c so that $a \mid bc$ but $a \nmid b$ and $a \nmid c$?*

Yes! For example, take $a = 6$, $b = 2$, and $c = 3$. Then $a \nmid b$ because $6 \nmid 2$, $a \nmid 3$ because $6 \nmid 3$, yet $a \mid bc$ because $6 \mid 2 \cdot 3$.

More generally, any time a is not prime, there will be integers b and c so that $a \mid bc$ but $a \nmid b$ and $a \nmid c$. Why should this be true?

4. (a) *Show that if $a \in \mathbb{Z}$, then $3 \mid a^3 - a$*

Suppose that $a \in \mathbb{Z}$. Consider $a^3 - a = a(a-1)(a+1)$.

By the Euclidean division algorithm, there exist $q, r \in \mathbb{Z}$ with $a = 3q + r$ where $0 \leqslant r < 3$. We consider the three possible values of r as three possible cases.

<u>Case 1:</u> $r = 0$

In this case, $a = 3q$ and hence,

$$a^3 - a = a(a-1)(a+1) = 3q(3q-1)(3q+1)$$

so $a^3 - a$ is a multiple of 3.

<u>Case 2:</u> $r = 1$

In this case, $a = 3q + 1$ and hence,

$$a^3 - a = (3q+1)(3q)(3q+2)$$

so $a^3 - a$ is a multiple of 3.

<u>Case 3:</u> $r = 2$

In this case, $a = 3q + 2$ and hence,

$$a^3 - a = (3q+2)(3q+1)(3q+3) = 3(3q+2)(3q+1)(q+1)$$

so $a^3 - a$ is a multiple of 3.
Since these are the only three possible values of r, we have shown that $a^3 - a$ is always a multiple of 3.

(b) *Show that if $a \in \mathbb{Z}$, then $5 \mid a^5 - a$*

Again, suppose that $a \in \mathbb{Z}$ and consider $a^5 - a = a(a-1)(a+1)(a^2+1)$.

By the Euclidean division algorithm, there exist $q, r \in \mathbb{Z}$ so that $a = 5q + r$ and $0 \leqslant r < 5$. We consider the possible values of r in the following cases:

<u>Case 1:</u> $r = 0, 1$, or 4.

In this case, either $a, a-1$, or $a+1$ is divisible by 5 and hence, $a^5 - a = a(a-1)(a+1)(a^2+1)$ is divisible by 5.

<u>Case 2:</u> $r = 2$

In this case, we claim that $5 \mid a^2 + 1$. Note that

$$a^2 + 1 = (5q+2)^2 + 1 = 25q^2 + 20q + 5 = 5(5q^2 + 4q + 1)$$

and hence, $a^2 + 1$ is divisible by 5. Therefore, $a^5 - a = a(a-1)(a+1)(a^2+1)$ is divisible by 5.

Case 3: $r = 3$

In this case, we claim again that $5 \mid a^2 + 1$. Note that

$$a^2 + 1 = (5q + 3)^2 = 25q^2 + 30q + 10 = 5(5q^2 + 6q + 2)$$

and hence, $a^2 + 1$ is divisible by 5. Therefore, $a^5 - a$ is divisible by 5.

Since these are the five possible values of r, we have shown that $a^5 - a$ is always a multiple of 5.