1. *Show that if a and b are positive integers, then there is a smallest positive integer of the form $a - bk$ for $k \in \mathbb{Z}$.*

   Let $S := \{a - bk > 0 \mid k \in \mathbb{Z}\}$. Since all elements of S are positive, S is a subset of $\mathbb{N}$. Moreover, S is nonempty because it contains the element $a - (-1)b$ which is positive because both a and b are positive. By the well-ordering principle of the natural numbers, S has a least element. This element is the smallest positive integer of the form $a - bk$ for $k \in \mathbb{Z}$.

2. *Prove that 7^n has the last two digits (in base 10)*

$$\begin{cases} 07 & \text{if } n \text{ is of the form } 4k+1 \\ 49 & \text{if } n \text{ is of the form } 4k+2 \\ 43 & \text{if } n \text{ is of the form } 4k+3 \\ 01 & \text{if } n \text{ is of the form } 4k \end{cases}$$

First, we want to observe the following fact: $2401 \cdot j$ has the same last two digits as j. This is because

$$2401 \cdot j = (24 \cdot 100 + 1) \cdot j = 24 \cdot 100j + j$$

and $24 \cdot 100j$ has the last two digits 00.

Now we proceed to proving the main claim by strong induction. Observe that

$$\begin{aligned} 7^0 &= 01 \\ 7^1 &= 07 \\ 7^2 &= 49 \\ 7^3 &= 343 \end{aligned}$$

so the claim is true for $n = 0, 1, 2,$ and 3. Suppose that the claim is true for $0, 1, 2, \ldots, n-1$. Then observe that $7^n = 2401 * 7^{n-4}$. Hence, we see that 7^n has the same last two digits as 7^{n-4}. By induction hypothesis, we know that 7^{n-4} has last two digits

$$\begin{cases} 07 & \text{if } n-4 \text{ is of the form } 4k+1 \\ 49 & \text{if } n-4 \text{ is of the form } 4k+2 \\ 43 & \text{if } n-4 \text{ is of the form } 4k+3 \\ 01 & \text{if } n-4 \text{ is of the form } 4k \end{cases}$$

But observe that if $n - 4 = 4k$, then $n = 4k + 4 = 4(k + 1)$, so n is of the same form. Likewise, if $n - 4 = 4k + 1$, then $n = 4k + 5 = 4(k + 1) + 1$, so n is again of the same form. If $n - 4 = 4k + 2$, then $n = 4k + 6 = 4(k + 1) + 2$ and if $n - 4 = 4k + 3$, then $n = 4k + 7 = 4(k + 1) + 3$. Hence, n is always of the same form as $n - 4$ and so we conclude that 7^n has last two digits

$$\begin{cases} 07 & \text{if } n \text{ is of the form } 4k+1 \\ 49 & \text{if } n \text{ is of the form } 4k+2 \\ 43 & \text{if } n \text{ is of the form } 4k+3 \\ 01 & \text{if } n \text{ is of the form } 4k \end{cases}$$

This completes the argument.

3. *What is wrong with the following proof?*

Proposition: All horses are the same color.

Proof. By (strong) induction on the number of horses.

Base cases: This is true if there are zero horses. It is also true if there is only one horse.

Inductive step: We assume that the statement holds for any group of k horses (or smaller) and show that it holds for a group of $k+1$ horses. Suppose we have a group of $k+1$ horses. Choose one, call it Winnie. The group, minus Winnie, has only k horses, so those horses are all the same color by assumption. Now choose another horse, call it Tigger. The group, minus Tigger (but including Winnie), has k horses again, and so they are all the same color by assumption. The overlap, $k-1$ horses, are also all of the same color by assumption. Therefore, any group of horses are the same color. Since there are a finite number of horses in the world, they must all be of the same color. $\square$

This argument fails because it doesn't prove enough base cases. It claims that "the overlap, $k-1$ horses, are also all of the same color by assumption" and uses this claim to argue that Tigger is the same color as the group of $k-1$ horses which are already the same color as Winnie. However, this requires the overlap to have a nonzero number of horses in it (i.e. that k is at least two)! But we can see that when $k=1$, there are two horses (Tigger and Winnie) and they certainly don't have to have the same color.

4. *Find three diferent formulas or rules for the terms of a sequence* $\{a_n\}$ *if the first three terms of the sequence are 1, 2, 4.*

One possible formula is $a_n = 2^n$ for $n \geqslant 0$. Another possible formula is $a_n = \frac{1}{2}n^2 + \frac{1}{2}n + 1$ for $n \geqslant 0$. Yet another is $a_n = n^3 - \frac{5}{2}n^2 + \frac{5}{2}n + 1$ for $n \geqslant 0$.

5. *Let H_n be the nth partial sum of the harmonic series. I.e.* $H_n = \sum_{j=1}^{n} \frac{1}{j}$

(a) *Prove that* $H_{2^n} \geqslant 1 + \frac{n}{2}$

We proceed by induction. Note that $H_{2^0} = 1 \geqslant 1 + \frac{0}{2}$, which is the base case.

Now suppose that $H_{2^n} \geqslant 1 + \frac{n}{2}$. Observe that

$$\begin{aligned} H_{2^{n+1}} &= \sum_{j=1}^{2^{n+1}} \frac{1}{j} \\ &= H_{2^n} + \sum_{j=2^n+1}^{2^{n+1}} \frac{1}{j} \end{aligned}$$

Now note that in the sum $\sum_{j=2^n+1}^{2^{n+1}} \frac{1}{j}$, $j \leqslant 2^{n+1}$ and so

$$\begin{aligned} H_{2^{n+1}} &= H_{2^n} + \sum_{j=2^n+1}^{2^{n+1}} \frac{1}{j} \\ &\geqslant H_{2^n} + \sum_{j=2^n+1}^{2^{n+1}} \frac{1}{2^{n+1}} \\ &= H_{2^n} + (2^{n+1} - 2^n)\frac{1}{2^{n+1}} \\ &= H_{2^n} + \frac{1}{2} \end{aligned}$$

By induction hypothesis, $H_{2^n} \geqslant 1 + \frac{n}{2}$ and so we conclude that

$$H_{2^{n+1}} \geqslant H_{2^n} + \frac{1}{2} \geqslant 1 + \frac{n}{2} + \frac{1}{2} = 1 + \frac{n+1}{2}$$

which is what we needed to verify.

(b) *Prove that* $H_{2^n} \leqslant 1 + n$

As in part (a), we proceed by induction. Note that $H_{2^0} = 1 \leqslant 1 + 0$, which is the base case.

Now we assume that $H_{2^n} \leqslant 1 + n$. Then

$$H_{2^{n+1}} = H_{2^n} + \sum_{j=2^n+1}^{2^{n+1}} \frac{1}{j}$$

In the sum $\sum_{j=2^n+1}^{2^{n+1}} \frac{1}{j}$, we note that $j > 2^n$ and so

$$
\begin{aligned}
H_{2^{n+1}} &= H_{2^n} + \sum_{j=2^n+1}^{2^{n+1}} \frac{1}{j} \\
&< H_{2^n} + \sum_{j=2^n+1}^{2^{n+1}} \frac{1}{2^n} \\
&= H_{2^n} + (2^{n+1} - 2^n)\frac{1}{2^n} \\
&= H_{2^n} + 1
\end{aligned}
$$

Under the induction hypothesis that $H_{2^n} \leqslant 1 + n$, we find that

$$H_{2^{n+1}} \leqslant H_{2^n} + 1 \leqslant 1 + (n+1)$$

which is what we needed to show.

(c) *Why does this imply that $H_n \approx \log_2(n)$?*

Letting $m = 2^n$, we can rewrite the inequality $1 + \frac{n}{2} \leqslant H_{2^n} \leqslant 1 + n$ as $1 + \frac{\log_2(m)}{2} \leqslant H_m \leqslant 1 + \log_2(m)$. Since H_m is trapped between two linear functions of $\log_2(m)$, it's reasonable to conclude that $H_m \approx \log_2(m)$.

Note that this allows us to prove that $\lim_{m\to\infty} H_m = \infty$. However, it's more interesting to note that H_m goes to infinity at a rate no faster than $\log_2(m)$.

Of course, this lines up with some geometric intuition that we have as well. H_m is a Riemann sum with m rectangles for $\int_1^m \frac{1}{x}\,dx$ and by the fundamental theorem of calculus, $\int_1^m \frac{1}{x}\,dx = \log m$.