1. Let $S = \{k > 0 : k \equiv 1 \mod 3\}$. Find an infinite product for the generating function of $p_S(n)$. Expand that product to find $p_S(n)$ for the first several n (up until $n = 9$ish)

   By Theorem 7.21, the generating function for $p_S(n)$ is

$$
\begin{aligned}
\sum_{n\in\mathbb{N}} p_S(n)x^n &= \prod_{j\in S} \frac{1}{1-x^j} \\
&= \prod_{k\in\mathbb{N}} \frac{1}{1-x^{1+3k}} \\
&= \prod_{k\in\mathbb{N}} \left(1 + x^{1+3k} + x^{2(1+3k)} + x^{3(1+3k)} + \cdots\right) \\
&= \left(1 + x + x^2 + \cdots\right)\left(1 + x^4 + x^8 + x^{12} + \cdots\right)\left(1 + x^7 + x^{14} + x^{21} + \cdots\right)\left(1 + x^{10} + x^{20} + \cdots\right)\cdots \\
&= 1 + x + x^2 + x^3 + 2x^4 + 2x^5 + 2x^6 + 3x^7 + 4x^8 + 4x^9 + \cdots
\end{aligned}
$$

   As a consequence, we have

x	0	1	2	3	4	5	6	7	8	9
$p_S(n)$	1	1	1	1	2	2	2	3	4	4

2. What is the generating function for $p(n \mid \text{parts are distinct powers of } 2)$?

   Let $S = \{2^k : k \in \mathbb{N}\}$. Then by theorem 7.21, the generating function for $p_S^D(n)$ is

$$\begin{aligned}\sum_{n\in\mathbb{N}} p_S^D(n)x^n &= \prod_{j\in S}(1+x^j)\\ &= \prod_{k\in\mathbb{N}}\left(1+x^{2^k}\right)\\ &= (1+x)(1+x^2)(1+x^4)(1+x^8)\cdots\end{aligned}$$

   Since every positive integer has a unique binary representation, we also have $p_S^D(n) = 1$ for all n. Hence,

$$1+x+x^2+x^3+\cdots = \sum_{n\in\mathbb{N}} p_S^D(n)x^n = (1+x)(1+x^2)(1+x^4)(1+x^8)\cdots$$

3. Show that if n is a positive integer, then

$$p(n) \leqslant \frac{p(n-1)+p(n+1)}{2}$$

Hint: It may be helpful to use the fact from last week's worksheet that $p(n) = p(n-1) + p_2(n)$. This fact (probably) won't be immediately helpful though...

First observe that the statement that

$$p(n) \leqslant \frac{p(n-1)+p(n+1)}{2}$$

is equivalent to the statement that $p(n) - p(n-1) \leqslant p(n+1) - p(n)$. By problem 5 on the week 3 group work, however, this is equivalent to showing that $p_2(n) \leqslant p_2(n+1)$. We show the latter claim now.

Let S_n be the set of all partitions of n all of whose parts are greater than or equal to 2 and let S_{n+1} be the set of all partitions of $n+1$ all of whose parts are greater than or equal to 2. Define a function

$$\begin{aligned} f : S_n &\to S_{n+1} \\ (\lambda_1, \ldots, \lambda_r) &\mapsto (\lambda_1 + 1, \lambda_2, \lambda_3, \ldots, \lambda_r) \end{aligned}$$

Since f is injective, we must have $p_2(n) = \#S_n \leqslant \#S_{n+1} = p_2(n+1)$, which proves the claim.