Section 7.2

Some props of φ:

- $\varphi : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$
- if $(m, n) = 1$, then $\varphi(mn) = \varphi(m)\varphi(n)$
- $\varphi(p^e)$ is easy to compute when p prime.

Def: An <u>arithmetic function</u> is a fn. defined on all positive integers

Def: An arithmetic function f is called <u>multiplicative</u> if $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$. f is <u>completely multiplicative</u> if $f(mn) = f(m)f(n)$ for all m, n.

Ex: $\varphi(n)$ is multiplicative

$\varphi(n)$ is not completely multiplicative

$\rightsquigarrow \varphi(4) \neq \varphi(2)\varphi(2)$

Def: The <u>sum of divisors</u> function is

$$\sigma_1 = \sigma : \mathbb{Z}_{>0} \longrightarrow \mathbb{Z}_{>0}$$
$$n \longmapsto \sum_{\substack{d \mid n \\ d > 0}} d$$

Bonus def:

$$\sigma_k(n) = \sum_{d \mid n} d^k$$

Def: The <u>number of divisors</u> function

$$\tau : \mathbb{Z}_{>0} \longrightarrow \mathbb{Z}_{>0}$$
$$n \longmapsto \sum_{\substack{d \mid n \\ d > 0}} 1$$

Ex:

n	1	2	3	4	5	6
$\sigma(n)$	1	3	4	7	6	12
$\tau(n)$	1	2	2	3	2	4

<u>$n=1$</u>

positive divisors. 1

<u>$n=2$</u>

pos. div.: 1, 2

<u>$n=3$</u>

pos. div: 1, 3

<u>$n=4$</u>

pos. div: 1, 2, 4

<u>$n=5$</u>

pos. div: 1, 5

<u>$n=6$</u>

pos div: 1, 2, 3, 6

Observations.

$$\sigma(6)=\sigma(2)\,\sigma(3)$$
$$\tau(6)=\tau(2)\,\tau(3)$$
$$\sigma(4)\neq\sigma(2)\,\sigma(2)$$
$$\tau(4)\neq\tau(2)\,\tau(2)$$

Thm: Suppose f is multiplicative. Then

$$F(n) = \sum_{d|n} f(d)$$ is also multiplicative

Note: if $f(x) = x$, $g(x) = 1$

f and g are multiplicative

$f(mn) = mn = f(m)\, f(n)$

$g(mn) = 1 = g(m)\, g(n)$

So by Thm, $\sigma(n) = \sum_{d|n} f(d)$

$\tau(n) = \sum_{d|n} g(d)$

are multiplicative

Also, $\varphi(n)$ is multiplicative

so $F(n) = \sum_{d|n} \varphi(d)$

Thm $\Rightarrow$ $F(n)$ is multiplicative

From 7.1: $F(n) = n$

Pf by example: $n = 36 = 4 \cdot 9$

Goal: $F(36) = F(4)\,F(9)$

$$F(n) = \sum_{d \mid n} f(d)$$

$$F(36) = \sum_{d \mid 36} f(d) =$$

$$= f(1) + f(2) + f(3) + f(4) + f(6) + f(9) + f(12) + f(18) + f(36)$$

$$= f(1\cdot 1) + f(2\cdot 1) + f(1\cdot 3) + f(4\cdot 1) + f(2\cdot 3) + f(1\cdot 9) + f(4\cdot 3) + f(2\cdot 9) + f(4\cdot 9)$$

$$= f(1)f(1) + f(2)f(1) + f(1)f(3) + f(4)f(1) + f(2)f(3) + f(1)f(9) + f(4)f(3) + f(2)f(9) + f(4)f(9)$$

$$= f(1)f(1) + f(1)f(3) + f(1)f(9) + f(2)f(1) + f(2)f(3) + f(2)f(9) + f(4)f(1) + f(4)f(3) + f(4)f(9)$$

$$= f(1)\left(f(1)+f(3)+f(9)\right) + f(2)\left(f(1)+f(3)+f(9)\right)$$
$$+ f(4)\left(f(1)+f(3)+f(9)\right)$$

$$= \left(f(1)+f(3)+f(9)\right)\left(f(1)+f(2)+f(4)\right)$$

$$= F(9)\,F(4)$$

Pf of Thm: Suppose $(m,n)=1$

Goal: $F(mn) = F(m)\,F(n)$

Since $(m,n)=1$, for every divisor

$d \mid mn$, $\exists!\, d_1, d_2 : d_1 \mid m,\ d_2 \mid n$

and $d = d_1 d_2$

$$F(mn) = \sum_{d \mid mn} f(d) = \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f(d_1 d_2)$$

$$= \sum_{\substack{d_1 | m \\ d_2 | n}} f(d_1) f(d_2)$$

$$= \sum_{d_1 | m} \sum_{d_2 | n} f(d_1) f(d_2)$$

$$= \sum_{d_1 | m} f(d_1) \sum_{d_2 | n} f(d_2)$$

$$= \sum_{d_1 | m} f(d_1) F(n)$$

$$= F(n) \sum_{d_1 | m} f(d_1)$$

$$= F(n) F(m)$$

p prime, $e \geq 1$

$\longrightarrow$ divisors of p^e: $1, p, p^2, \dots, p^e$

$$\tau(p^e) = e+1$$

$$\sigma(p^e) = 1 + p + p^2 + \cdots + p^e = \frac{p^{e+1} - 1}{p-1}$$

Ex: $\tau(2^3 \cdot 5^2) = \tau(2^3)\,\tau(5^2) = (3+1)(2+1)$
$= 12$

$\sigma(2^3 \cdot 5^2) = \sigma(2^3)\,\sigma(5^2) = \frac{2^4 - 1}{2-1} \cdot \frac{5^3 - 1}{5-1}$
$= 465$