Homework 4

1. *For a positive integer n, show that*

$$\tau(n^2) = \#\{(a,b) \in \mathbb{Z}^2 : a,b > 0 \textit{ and } \operatorname{lcm}(a,b) = n\}$$

Your answer here...

2. *A partition of n is said to be self-conjugate if it is its own conjugate.*

   (a) *Suppose that $(\lambda_1, \ldots, \lambda_r)$ is a partition of n. Let $S = \{\lambda_j : \lambda_j \geq j\}$. Show that if $\lambda_k \in S$, then $\lambda_{k-1} \in S$. Conclude that S has the form $\{\lambda_1, \ldots, \lambda_t\}$ for some t.*

   Your answer here...

   (b) *Let $(\lambda_1, \ldots, \lambda_r)$ be a self-conjugate partition of n. Suppose that a dot in the Ferrers diagram of $(\lambda_1, \ldots, \lambda_r)$ is in row j and column ℓ. Show that either $\lambda_j \geq j$ or $\lambda_\ell \geq \ell$.*

   Your answer here...

(c) *Let* $(\lambda_1, \ldots, \lambda_r)$ *be a self-conjugate partition of* n. *Let* $k = \#\{j : \lambda_j \geq j\}$. *For each* $1 \leq i \leq k$, *define* $\rho_i = 2\lambda_i - (2i - 1)$. *Prove that* $(\rho_1, \ldots, \rho_k)$ *is a partition of* n *with distinct, odd parts.*

Your answer here...

(d) *Let* O *be the set of odd positive integers. Show that the number of self-conjugate partitions of* n *is equal to* $p_O^D(n)$.

Your answer here...

3. *Use the previous problem to show that $p(n)$ is odd if and only if $p_O^D(n)$ is odd.*

Your answer here...

4. *(Extra Credit) Use Ferrers diagrams to show that* $p^D(n)$ *is equal to the number of partitions of the form* $(\lambda_1, \ldots, \lambda_k)$ *where for each* i *so that* $1 \leq i \leq \lambda_1$*, there exists a* j *so that* $i = \lambda_j$*.*