Section 1.1: Numbers and Sequences

Easiest number: 0

Create a fn. called "Successor"

$\longrightarrow s(n) = n+1$

0 + operation of $s(\cdot) \rightsquigarrow 0, 1, 2, 3, \ldots\ldots$

Def: The set of natural numbers is

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

blackboard bold

\mathbb{N} ⇐ LaTeX

$\mathbb{N}$ has: addition, multiplication, exponentiation

$\mathbb{N}$ doesn't have: subtraction, division

$\mathbb{N}$ is what number theorests study!

To get subtraction: create "additive inverses"

x is an additive inverse for n if $x+n=0$

Def: $\mathbb{Z} = \{\ldots\ldots, -2, -1, 0, 1, 2, 3, \ldots\ldots\}$ is the set of integers

$\mathbb{Z}$ has: addition, exp., sub.

$\mathbb{Z}$ does not have: division

Def: $\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \} = \mathbb{Q}$

is the set of rational numbers

$\mathbb{Q}$ has: $+, -, \cdot, \div$

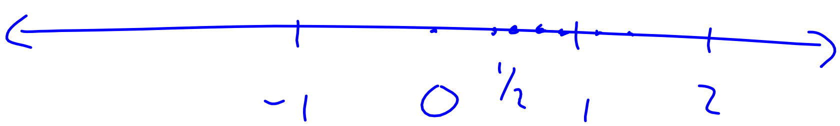
↳ $\mathbb{Q}$ is a field!

Question. Does $\mathbb{Q}$ contain all of the real numbers?

$\mathbb{R}$

↓

real numbers



Answer: No!

Claim: $\sqrt{2}$ is not rational

Proof: Suppose, by ↯ (contradiction), that $\sqrt{2}$ is rational.

Then $\sqrt{2} = \frac{a}{b}$ where $a, b \in \mathbb{Z}_{>0}$ or $\mathbb{Z}^+$ (positive integers)

Define $S := \{ k\sqrt{2} : k, k\sqrt{2} \in \mathbb{Z}_{>0} \}$

Note: $S \neq \emptyset$ (\var nothing)

because $b\sqrt{2} = b \cdot \frac{a}{b} = a \in \mathbb{Z}_{>0}$

and $b \in \mathbb{Z}_{>0}$

$\Rightarrow b\sqrt{2} \in S \Rightarrow S \neq \emptyset$

Property (Well-ordering): Any nonempty $X \subseteq \mathbb{N}$ has a (unique) least element

Note: $S \neq \emptyset$, $S \subseteq \mathbb{N}$

Therefore, S has a least element

Suppose $s = t\sqrt{2}$ is the least element

Goal: find a smaller member of S

① $(s-t)\sqrt{2} \in S$

$s\sqrt{2} - t\sqrt{2} = 2t - s \in \mathbb{Z}$

$$s\sqrt{2} - t\sqrt{2} = s\sqrt{2} - s > 0 \quad (\text{b/c } \sqrt{2} > 1 \rightarrow s\sqrt{2} > s)$$

NTS: $s - t \in \mathbb{Z}_{>0}$

$s - t \in \mathbb{Z}$ ✓

$s = t\sqrt{2} \to s > t$

$(s-t)\sqrt{2} \in S$ ✓

② $(s-t)\sqrt{2}$ is smaller than s

Note: $(s-t)\sqrt{2} = s\sqrt{2} - t\sqrt{2}$

$= s\sqrt{2} - s$

$= s(\sqrt{2} - 1)$

$< s$

because $\sqrt{2} - 1 < 1$

Contradiction! (s was the least elt. of S)

(element)

Therefore, $\sqrt{2}$ is not rational

Lesson: $\mathbb{R}$ is bigger than $\mathbb{Q}$

Fact: π, e are also irrational

$0 \in \mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$

Observe: $\sqrt{2}$ is the root of a polynomial with integer coefficients. $x^2 - 2$

Fact: π, e do not have this property

Def: The number α is <u>algebraic</u> if it is a root of some polynomial with integer coefficients. α is <u>transcendental</u> otherwise.

Let $\overline{\mathbb{Q}} := \{\alpha : \alpha \text{ is algebraic}\}$

\bar{\mathbb{Q}}

\overline{\mathbb{Q}}

Note: i is algebraic: i is a root of x^2+1

$i \notin \mathbb{R}$

$$0 \in \mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$$

$$\overline{\mathbb{Q}} \subsetneq \mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}\}$$

($\mathbb{Q} \subsetneq \overline{\mathbb{Q}}$, $\mathbb{R} \subsetneq \mathbb{C}$)

Q: How do we know that $\overline{\mathbb{Q}}$ doesn't contain $\mathbb{R}$?

<u>Sequences</u>

Def: A <u>sequence</u> is a list of numbers

$a_0, a_1, a_2, a_3, \ldots$

Ex: Create a formula for $\underset{a_0}{3}, \underset{a_1}{11}, \underset{a_2}{19}, 27, 35, \ldots$

$+8 \quad +8$

$$a_n = 3 + 8n$$

Def: A sequence of the form

$$a,\ a+d,\ a+2d, \ldots,\ a+nd, \ldots$$

is called an <u>arithmetic progression</u>

AIR - ith - metic

Fact: $a_{n+1} - a_n$ is constant in an arithmetic progression

Def: A sequence of the form

$$a,\ a\cdot r,\ a\cdot r^2,\ a\cdot r^3, \ldots,\ a\cdot r^n, \ldots$$

is called a <u>geometric progression</u>

Ex: $1, 2, 4, 8, 16, \ldots$

Set sizes

Def: A set S is <u>countable</u> if it is finite OR there exists a function $f: \mathbb{N} \to S$ which is one-to-one and onto (i.e. f is a <u>bijection</u>)

Recall: $f: X \to Y$ is <u>one-to-one</u> (or <u>injective</u>) if for all $x_1, x_2 \in X$

$$f(x_1) = f(x_2) \to x_1 = x_2$$

$$\left(x_1 \neq x_2 \to f(x_1) \neq f(x_2) \right.$$

"unique inputs have unique outputs")

Recall: $f: X \to Y$ is <u>surjective</u> (or <u>onto</u>) if for all $y \in Y$, there exists $x \in X$ for which $f(x) = y$

Observe:

An infinite set S is countable if and only if S can be written as a sequence

Reason:
$$\begin{array}{cccccc} S: & a_0 & a_1 & a_2 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \\ \mathbb{N}: & 0 & 1 & 2 & \cdots \end{array}$$

Claim: $\mathbb{Z}$ is countable

Pf: $0, 1, -1, 2, -2, 3, -3, \ldots$

Claim: $\mathbb{Q}$ is countable

Pf:
$$\begin{array}{ccccccc} 0/1 \rightarrow & 1/1 & -1/1 \rightarrow & 2/1 & -2/1 & 3/1 & -3/1 \\ 0/2 & 1/2 & -1/2 & 2/2 & -2/2 & 3/2 & -3/2 \\ 0/3 & 1/3 & -1/3 & 2/3 & -2/3 & 3/3 & -3/3 \cdots \\ \vdots & & & & & & \end{array}$$

Fact: $\overline{\mathbb{Q}}$ is countable

Claim: $\mathbb{R}$ is uncountable

Pf: By $\lightning$ assume $\mathbb{R}$ is countable

Then we can write $\mathbb{R}$ as a sequence

In particular: $S = \{x \in \mathbb{R} : 0 \leq x < 1\}$

can be written as a sequence

$$
\begin{array}{l}
0.\,a_{00}\ a_{01}\ a_{02}\ a_{03}\ \cdots\cdots \\
0.\,a_{10}\ a_{11}\ a_{12}\ a_{13}\ \cdots\cdots \\
0.\,a_{20}\ a_{21}\ a_{22}\ a_{23}\ \cdots\cdots \\
\vdots \qquad\qquad\qquad\qquad \ddots
\end{array}
$$

$x \stackrel{?}{=}$

Define: $b_n = \begin{cases} 1 & \text{if } a_{nn} \neq 1 \\ 2 & \text{if } a_{nn} = 1 \end{cases}$

Note: $b_n \neq a_{nn}$ for all n

$x = 0.\,b_0\, b_1\, b_2\, b_3 \cdots\cdots$

x not in our sequence!