Section 7.5 – Partitions I

- How many ways can you add numbers to get 10?
  - if "numbers" included negatives, we'd always get inf. many ways

$$10 = -1 + 11$$
$$= -1 + -1 + 12$$
$$= \vdots$$

- How many ways can you add ~~numbers~~ pos. integers to get 10?

$$10 = 4+6$$
$$= 5+5$$
$$= 2+2+6$$
$$= 1+1+1+\cdots+1$$
$$= 6+2+2$$
$$= \vdots$$

the same

- Convention: Write down "ways" in non increasing order

Def: Given $n \in \mathbb{Z}_{>0}$, a <u>partition</u> of n is a tuple $(\lambda_1, \dots, \lambda_r)$ where $\lambda_1, \dots, \lambda_r \in \mathbb{Z}_{>0}$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$, and $\sum_{i=1}^{r} \lambda_i = \lambda_1 + \lambda_2 + \dots + \lambda_r = n$.

Each λ_i is called a <u>part</u>

↳ Note, this is not the same thing as a partition of a set

Ex: $(3,1,1)$ is a partition of 5

Nonex: $(1,3,1)$ is not a partition

Def: The <u>partition (counting) function</u> is

$p: \mathbb{N} \longrightarrow \mathbb{Z}_{>0}$ s.t. <u>$p(0) = 1$</u> (good) convention

and $p(n) =$ the number of partitions of n when $n > 0$.

Ex: $p(4)$

Partitions of 4.

(4)

$(3, 1)$

$(2, 2)$

$(2, 1, 1)$

$(1, 1, 1, 1)$

$\longrightarrow p(4) = 5$

- $p(n)$ is one of the most complicated functions in math!

Restricted Partitions

Q: How many partitions of 5 have only odd parts?

$$5 = 5$$
$$= 3 + 1 + 1$$
$$= 1 + 1 + 1 + 1 + 1$$

So 3

Def: Let $S \subseteq \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}_{>0}$

$p_S(n)$ = # of partitions of n into parts from S

not an exponent!

$p^D(n)$ = # of partitions of n into distinct parts

$p_m(n)$ = # of partitions of n into parts $\geq m$

↳ can combine these... $p_S^D(n)$, $p_{S,m}(n)$

↳ also $p(n \mid [\text{conditions here}])$

e.g. $p(n \mid \text{no part appears more than twice})$

$O = \{n \in \mathbb{Z}_{>0} : n \text{ is odd}\}$

$\rightarrow p_O(5)$

Ferrers Diagrams

Def: For any partition $(\lambda_1, \dots, \lambda_r)$, define the <u>Ferrers Diagram</u> to have r rows of dots so that in row j, there are λ_j dots

Ex: $(4, 4, 2, 1)$

- # of rows = # of parts = 4
- row 1 has $\lambda_1 = 4$ dots
- row 2 has $\lambda_2 = 4$ dots
- row 3 has $\lambda_3 = 2$ dots
- row 4 has $\lambda_4 = 1$ dot

```
*  *  *  *
*  *  *  *
*  *
*
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all rows → left-justified

Ex: (3, 2, 2, 2, 2)

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*  *  *
*  *
*  *
*  *
*  *
```

Observation

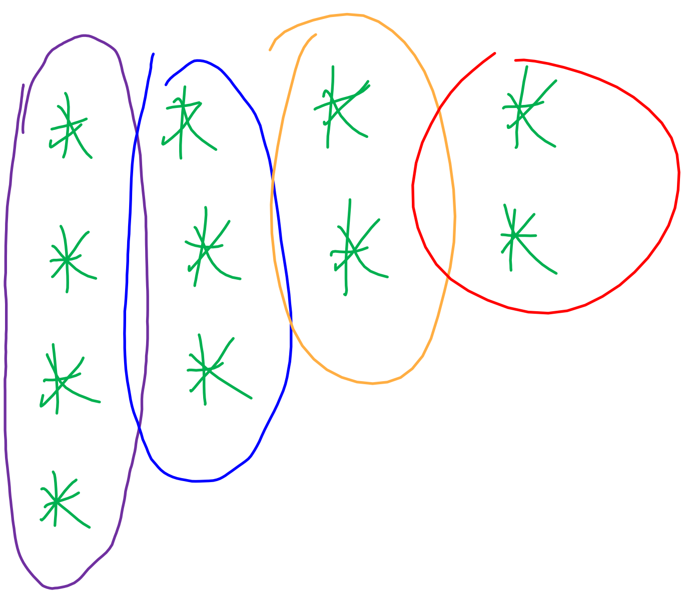
- # of rows = # of parts
- # of dots = number being partitioned

- # of columns
  $= \text{largest part}, \lambda_1$

- # of dots per row
  $= \lambda_i$

- # of dots per column?

Ex: $(4, 4, 2, 1)$

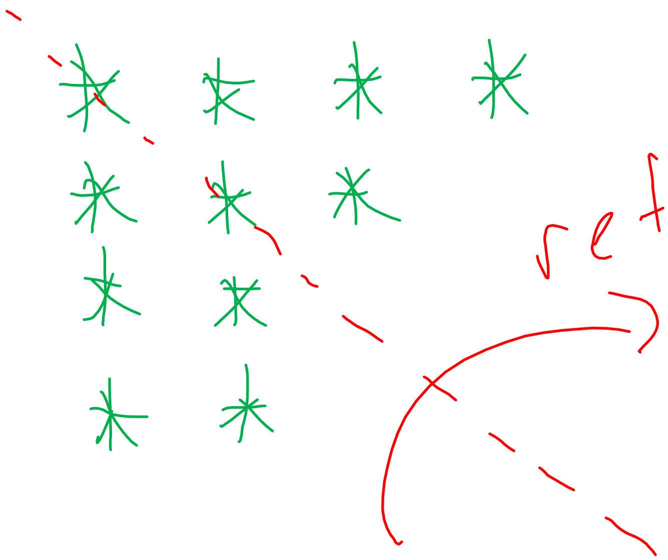


- # of dots in col. $\underline{1}$ = # of parts = 4
- # of dots in col 2 = # of parts = 3
  with size ≥ 2

- # of dots in col. 3 = # of parts with size ≥ 3 = 2

- # of dots in col. 4 = # of parts with size ≥ 4 = 2

$\rightsquigarrow$ new partition of 11: $(4, 3, 2, 2)$



reflect to get original FD.

Def: Given a partition $(\lambda_1, \ldots, \lambda_k)$, define the <u>conjugate partition</u> by $(\lambda_1', \ldots, \lambda_{\lambda_1}')$ so that for each $1 \leq i \leq \lambda_1$

$$\lambda_i' := \#\{\lambda_j : \lambda_j \geq i\}$$

$(= \#$ of dots in col. $i)$

Thm: The conjugate partition of $(\lambda_1, \dots, \lambda_r)$ is a partition of the same number

Pf: First, we show that

$$\sum_{i=1}^{r} \lambda_i = \sum_{j=1}^{\lambda_1} \lambda_j'$$

Follows from the Ferrers Diagrams, since the FD of $(\lambda_1', \dots, \lambda_{\lambda_1}')$ is the reflection of $(\lambda_1, \dots, \lambda_r)$

Next, we show $\lambda_1' \geq \lambda_2' \geq \dots \geq \lambda_{\lambda_1}'$

$$\begin{array}{ccccccc}
\{\lambda_j : \lambda_j \geq 1\} & \supseteq & \{\lambda_j : \lambda_j \geq 2\} & \cdots & \supseteq & \{\lambda_j : \lambda_j \geq \lambda_1\} \\
\big\downarrow \text{size} & & \big\downarrow \text{size} & & & \big\downarrow \text{size} \\
\lambda_1' & \geq & \lambda_2' & \geq \cdots & \geq & \lambda_{\lambda_1}'
\end{array}$$

Thm: If $n \in \mathbb{Z}_{>0}$, then (the number of partitions of n with largest part equal to r) is equal to (the number of partitions with exactly r parts)

Pf: Let S be the set of partitions of n with largest part equal to r

Let T be the set of partitions of n with exactly r parts

Define: $c : T \to S$

$$(\lambda_1, \ldots, \lambda_r) \mapsto (\lambda_1', \ldots, \lambda_{\lambda_1}')$$

Need to check: $(\lambda'_1, \dots, \lambda'_{\lambda_1}) \in S$

The largest part of $(\lambda'_1, \dots, \lambda'_{\lambda_1})$ is

$$\lambda'_1 = \#\{\lambda_j : \lambda_j \geq 1\} = r \checkmark$$

Note that c reflects FDs and hence is its own inverse.

So c is a bijection and

$$|T| = |S|$$

Ex: Find the number of partitions of 10 with largest part equal to 2.

By Thm $\longrightarrow$ can count # of partitions of 10 with 2 parts.

$$10 = \underbrace{9+1 = 8+2 = 7+3 = 6+4 = 5+5}_{5}$$

$$10 = 2+2+2+2+2$$
$$= 2+2+2+2+1+1$$
$$= 2+2+2+1+1+1+1$$
$$= 2+2+1+1+1+1+1+1$$
$$= \_ \ \_ \ \_ \ \_$$

Generating Functions

- Motivation:

Recall the binomial formula

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

$$(x+1)^n = \sum_{i=0}^{n} \binom{n}{n-i} x^i$$

why?

$$(x+1)(x+1)\cdots(x+1) = x^n + n\,x^{n-1} + \binom{n}{2}x^{n-2} + \cdots + \binom{n}{i}x^{n-i}$$

Ex:

$$\sum_{i=0}^{n}\binom{n}{i}x^i y^{n-i} = (x+y)^n = (y+x)^n = \sum_{i=0}^{n}\binom{n}{i}y^i x^{n-i}$$

$$\overset{(j=n-i)}{=} \sum_{j=0}^{n}\binom{n}{n-j}y^{n-j}x^j$$

$$\longrightarrow \binom{n}{i} = \binom{n}{n-i}$$

<u>Philosophy</u>

Sequences $\longrightarrow$ power series

Combinatorial $\longrightarrow$ analytic

Ex: $a_n = 2^n$ for $n \geq 0$

$$\rightarrow \quad f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2^n x^n$$

$$= \sum_{n=0}^{\infty} (2x)^n = \frac{1}{1-2x}$$

(when $|x| < \frac{1}{2}$)

Use Taylors Theorem $\rightarrow$ express a_n in terms of $f^{(n)}(0)$

If we know things about $f^{(n)}(x)$, then we learn things about a_n

Def: For a sequence $\{a_n\}_{n=0}^{\infty}$, the <u>generating function</u> for a_n is

$$\sum_{n=0}^{\infty} a_n x^n$$

Ex: The generating fn. for $\{p(n)\}_{n=0}^{\infty}$

is

$$p(0) + p(1)x + p(2)x^2 + p(3)x^3 + \cdots$$
$$= 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots$$

Thm: The generating function for $p(n)$ is $\displaystyle\prod_{j=1}^{\infty} \frac{1}{1-x^j} = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \frac{1}{1-x^3} \cdots$

Concern: What about convergence?

Ex: For <u>any</u> real x, there exists a way to rearrange the terms of the sequence $\frac{-1}{1}, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \ldots, \frac{(-1)^n}{n}, \ldots$

let's say that the rearrangement is $a_0, a_1, a_2, \ldots$ so that $x = \sum_{n=0}^{\infty} a_n$

Moral: There's some concern, but no problems that arise here

Pf of Thm:

$$\frac{1}{1-x} = 1+x+x^2+\cdots = \sum_{n=0}^{\infty} x^n$$

So $$\frac{1}{1-x^j} = 1+x^j+x^{2j}+x^{3j}+\cdots = \sum_{n=0}^{\infty} x^{jn}$$

$$\prod_{j=1}^{\infty} \frac{1}{1-x^j} = \prod_{j=1}^{\infty} \sum_{n=0}^{\infty} x^{jn}$$

$$= \prod_{j=1}^{\infty} \left(1+x^j+x^{2j}+x^{3j}+\cdots\right)$$

$$= \left(1+x+x^2+x^3+\cdots\right)\left(1+x^2+x^4+x^6+\cdots\right)\left(1+x^3+x^6+x^9+\cdots\right)$$

$$= 1+1\cdot x^1+2x^2+3x^3+\cdots$$

$$= p(0)+p(1)x+p(2)x^2+p(3)x^3+\ ???$$

Ways of getting x^3

- x^3 from first term, 1s from all else
- ~~x^2 from first term,~~

- x from first term, x^2 from second term, 1s from all else
- 1 from first term, 1 from second term, x^3 from 3^{rd} term, 1s from all else

Perspective:

$1 + x + x^2 + x^3 + \cdots$: exp. counting the number of 1s in a partition

$1 + x^{1\cdot 2} + x^{2\cdot 2} + x^{3\cdot 2} + \cdots$: exp. counts # of 2s in a partition

$\downarrow$

3 2s in a partition

$\vdots$

$1 + x^{1j} + x^{2j} + x^{3j} + \cdots$: exp counts # of js in a partition.

$$(1+x+x^2+\dots)(1+x^2+x^4+\dots)(1+x^3+x^6+\dots)\cdots$$

Coeff on x^n is # of ways

of writing $n = k_1 \cdot 1 + k_2 \cdot 2 + \dots + k_n n$

where $k_1 \geq 0,\ k_2 \geq 0, \dots,\ k_n \geq 0$

= # of ways of writing

$$n = \underbrace{1+1+\dots+1}_{k_1 \text{ times}} + \underbrace{2+2+\dots+2}_{k_2 \text{ times}} + \dots + \underbrace{n+\dots+n}_{k_n \text{ times}}$$

= # of partitions of n

$$G_0 \quad \prod_{j=1}^{\infty} \frac{1}{1-x^j} = (1+x+x^2+\dots)(1+x^2+x^4+\dots)\cdots$$

$$= \sum_{n=0}^{\infty} p(n) x^n \quad \checkmark$$

Ex / Thm: The generating fn. for $p^D(n)$ (the # of partitions of n into distinct parts, $(\lambda_1, \ldots, \lambda_r)$ s.t. $\lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_r$)

is

$$\sum_{n \in \mathbb{N}} p^D(n) x^n = \prod_{j=1}^{\infty} (1+x^j)$$

"Pf:" $\prod_{j=1}^{\infty} (1+x^j) = (1+x)(1+x^2)(1+x^3)(1+x^4)\cdots$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

a_0 = # of ways to pick a term from each binomial and multiply them to get 1

= 1 (pick all 1s)

a_1 = # of ways to pick a term from each binomial and multiply them to get x

$= 1$

$a_2 =$ " ... " to get x^2

$= 1$

$a_3 =$ " ... " to get x^3

$= 2$ (pick x, x^2, 1s

OR $1, 1, x^3$, 1s)

$a_n =$ " ... " to get x^n

$=$ number of ways
to pick distinct
#s $< n$ and add
them up to get n

$$= p^D(n)$$

Generalize: if $S \subseteq \mathbb{N}$, the gen. fns. for $p_S(n)$ and $p_S^D(n)$

(pts. from S ... distinct pts. from S)

are

$$\sum_{n \in \mathbb{N}} p_S(n) x^n = \prod_{j \in S} \frac{1}{1-x^j}$$

$$\sum_{n \in \mathbb{N}} p_S^D(n) x^n = \prod_{j \in S} (1+x^j)$$

Thm (Euler Parity): For every pos. int. n, $p_O(n) = p^D(n)$ $\quad (O = \{n \in \mathbb{Z} : n \text{ odd}\})$

i.e. # of partitions of n into odd pts

= # of partitions of n into distinct pts.

Pf: Goal: prove that

$$\sum_{n \in \mathbb{N}} p_o(n) x^n = \sum_{n \in \mathbb{N}} p^D(n) x^n$$

$$\sum_{n \in \mathbb{N}} p^D(n) x^n = \prod_{j=1}^{\infty} (1 + x^j)$$

$$\overset{*}{=} \prod_{j=1}^{\infty} \frac{1 - x^{2j}}{1 - x^j}$$

$*\ (1-x^j)(1+x^j) = 1 - x^{2j}$

$$= \frac{1-\cancel{x^3}}{1-x} \cdot \frac{\cancel{1-x^4}}{\cancel{1-x^2}} \cdot \frac{\cancel{1-x^6}}{1-x^3} \cdot \frac{\cancel{1-x^8}}{\cancel{1-x^4}} \cdots$$

$$= \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots$$

$$= \prod_{\substack{j \in O \\ j > 0}} \frac{1}{1-x^j} = \sum_{\substack{n \in O \\ n > 0}} p_o(n) x^n$$

Pentagonal # Thm

- $f(x) = \sum_{n \in \mathbb{N}} p(n) x^n = \prod_{j=1}^{\infty} \frac{1}{1-x^j}$

- Q: What function should I multiply $f(x)$ by to get 1?

- A: $\frac{1}{f(x)} = \prod_{j=1}^{\infty} 1-x^j$

- New Q: Does $\prod_{j=1}^{\infty} 1-x^j$ carry interesting combinatorial info?

Thm: $\prod_{j=1}^{\infty} (1-x^j) = \sum_{n \in \mathbb{N}} a_n x^n$ implies

$$a_n = p(n \mid \text{even number of distinct pts}) - p(n \mid \text{odd number of distinct pts})$$

Pf: $\prod_{j=1}^{\infty}(1-x^j) = (1-x)(1-x^2)(1-x^3)\cdots$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$a_0 = 1$

$a_1 = -1$

choose $-x$, $-x^2$

$a_2 = -1$

choose $-x^3$

$a_3 = 1 - 1 = 0$

choose $-x$, $-x^3$

$a_4 = 1 - 1 = 0$

choose $-x^4$

choose $-x^5$

choose $-x^4$, $-x$

$a_5 = -1 + 1 + 1$

choose $-x^2$, $-x^3$

$a_n = -p(n \mid \text{odd distinct pts})$

$+ p(n \mid \text{even distinct pts})$.

Thm (Euler's Pentagonal # Thm):
The # of partitions of n into an odd # of distinct pts. equals the # of partitions of n with an even # of distinct pts. unless $n = \frac{k(3k \pm 1)}{2}$ for some pos. int k.

If $n = \frac{k(3k \pm 1)}{2}$, then

$$p(n \mid \text{even \# of distinct pts}) - p(n \mid \text{odd \# of distinct pts}) = (-1)^k$$

In short:

$$p(n \mid \text{even \# of distinct pts}) - p(n \mid \text{odd \# of distinct pts}) = \begin{cases} (-1)^k & \text{if } n = \frac{(3k \pm 1)k}{2} \\ 0 & \text{else} \end{cases}$$

$$\text{So } \frac{1}{f(x)} = \prod_{j=1}^{\infty} (1 - x^j)$$

$$= \sum_{n \in \mathbb{N}} a_n x^n$$

$$= 1 + \sum_{k=1}^{\infty} (-1)^k x^{\frac{(3k-1)k}{2}} + (-1)^k x^{\frac{(3k+1)k}{2}}$$

$$= 1 + \sum_{k=1}^{\infty} (-1)^k x^{\frac{(3k-1)k}{2}} (1 + x^k)$$

$$f(x) = \sum_{n=0}^{\infty} p(n) x^n$$

$$1 = f(x) \cdot \frac{1}{f(x)}$$

$$= \left(p(0) + p(1)x + p(2)x^2 + p(3)x^3 + \cdots\right)$$
$$\left(1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots\right)$$

$$= p(0) + \big(p(1) - p(0)\big)x + \big(p(2) - p(1) - p(0)\big)x^2$$
$$+ \big(p(3) - p(2) - p(1)\big)x^3 + \big(p(4) - p(3) - p(2)\big)x^4$$
$$+ \big(p(5) - p(4) - p(3) + p(0)\big)x^5 + \cdots\cdots$$

$$p(0) = 1$$
$$p(1) - p(0) = 0 \longrightarrow p(1) = p(0) = 1$$
$$p(2) - p(1) - p(0) = 0 \longrightarrow p(2) = p(1) + p(0)$$
$$= 2$$
$$p(3) - p(2) - p(1) = 0$$
$$p(4) - p(3) - p(2) = 0$$
$$p(5) - p(4) - p(3) + p(0) = 0$$
$$\vdots$$
$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7)$$
$$- p(n-12) - p(n-15)$$

$$+ \cdots + (-1)^k p\left(n - \frac{k(3k-1)}{2}\right)$$

$$+ (-1)^k p\left(n - \frac{k(3k+1)}{2}\right)$$

$$+ \cdots = 0$$

Def: Euler's Partition Formula

Pentagonal #s: