1. Given that $13 = 3^2 + 2^2$, $29 = 5^2 + 2^2$, and $50 = 7^1 + 1^2$, write each of the following integers as the sum of two squares:

   (a) $377 = 13 \cdot 29$

   (b) $650 = 13 \cdot 50$

   (c) $18850 = 13 \cdot 29 \cdot 50$

2. Determine whether each of the following integers can be written as the sum of two squares:

   (a) 19

   (b) 29

   (c) 99

   (d) 999

   (e) 1000

   (f) 80

3. Show that the positive integer $n$ is not the sum of three squares of integers if $n = 8k + 7$ for some integer $k$

4. Show that every integer $n \geqslant 170$ is the sum of five positive squares.
*Hint:* $169 = 13^2 = 12^2 + 5^2 = 12^2 + 4^2 + 3^2 = 10^2 + 8^2 + 2^2 + 1^2$

5. Show Thue's lemma: If $p$ is prime and $a \not\equiv 0 \mod p$, then there exist integers $x, y$ so that $ax \equiv y \mod p$ with $0 < |x| < \sqrt{p}$ and $0 < |y| < \sqrt{p}$.
   *Hint:* Use the pigeonhole principle to show that there are two integers of the form $au - v$ with $0 \leqslant u < \sqrt{p}$ and $0 \leqslant v < \sqrt{p}$ that are congruent mod $p$. Construct $x$ and $y$ from the two values of $u$ and $v$.
   *Consider:* How does this give a "small fractional representation" of $a$ modulo $p$?

6. Use the previous problem to show that if $p$ is prime and $p \not\equiv 3 \mod 4$, then $p$ can be written as the sum of two squares.
   *Hint:* There is an $a$ so that $a^2 \equiv -1 \mod p$. Apply Thue's lemma to that value of $a$.
   *Note:* This is the theorem we skipped in class.