1. *Define Liouville's function* $\lambda(n)$ *so that* $\lambda(1) = 1$ *and for* $n \geqslant 2$, $\lambda(n) = (-1)^{e_1+\cdots+e_g}$ *when the prime factorization of* n *is* $p_1^{e_1} \cdots p_g^{e_g}$. *Is* λ *multiplicative? Is* λ *completely multiplicative?*

   We claim that λ is completely multiplicative. To see this, pick any $a, b \in \mathbb{Z}_{>0}$. Without loss of generality, we can write

$$a = p_1^{a_1} \cdots p_g^{a_g}$$
$$b = p_1^{b_1} \cdots p_g^{b_g}$$

   for distinct primes $p_1, \ldots, p_g$ and exponents $a_1, \ldots, a_g, b_1, \ldots, b_g \geqslant 0$. Note that $\lambda(a) = (-1)^{a_1+\cdots+a_g}$ even if some of the a_i are 0 (and we apply similar reasoning in the remainder of the problem). Hence,

$$\begin{aligned}
\lambda(ab) &= \lambda\left(p_1^{a_1+b_1} \cdots p_g^{a_g+b_g}\right) \\
&= (-1)^{a_1+b_1+\cdots+a_g+b_g} \\
&= (-1)^{a_1+a_2+\cdots+a_g} \cdot (-1)^{b_1+b_2+\cdots+b_g} \\
&= \lambda(a) \cdot \lambda(b)
\end{aligned}$$

   Since $\lambda(ab) = \lambda(a)\lambda(b)$ for all positive a, b, we conclude that λ is completely multiplicative. In particular, this implies that λ is also multiplicative.

2. *An arithmetic function f is said to be* <u>additive</u> *if $f(mn) = f(m) + f(n)$ for all relatively prime positive integers m and n. f is said to be* <u>completely additive</u> *if $f(mn) = f(m) + f(n)$ for all positive integers m and n. For any prime integer p, define the function $v_p(n)$ by defining*

$$v_p(n) := \max\{k \in \mathbb{N} : p^k \mid n\}$$

   (a) *Is v_p additive? Is it completely additive?*

   We first claim that $v_p(m) = k$ if and only if $m = p^k \cdot \ell$ for some $\ell \in \mathbb{Z}$ where $(\ell, p) = 1$. If $v_p(m) = k$, then $p^k \mid m$, but $p^{k+1} \nmid m$. So there exists $\ell \in \mathbb{Z}$ so that $m = p^k \ell$. If $p \mid \ell$, then $p^{k+1} \mid m$, which would be a contradiction. Hence, $(p, \ell) = 1$. For the converse, if $m = p^k \ell$ where $(\ell, p) = 1$, we see that p^{k+1} cannot divide m: if it did, then $p^{k+1} \mid p^k \ell$ and so $p \mid \ell$, which would contradict the fact that $(\ell, p) = 1$. Therefore, $k = \max\{n : p^n \mid m\} = v_p(m)$.

   With that technical point out of the way, we proceed to prove that v_p is completely additive. Suppose that a, b are positive integers with $v_p(a) = k$ and $v_p(b) = \ell$. Write $a = p^k m$ for some $m \in \mathbb{Z}$ with $(m, p) = 1$ and $b = p^\ell n$ for some $n \in \mathbb{Z}$ with $(n, p) = 1$. Then $ab = p^{k+\ell} mn$. Note that $(p, mn) = 1$ because p is prime and $p \nmid m, n$. Hence, $v_p(ab) = k + \ell = v_p(a) + v_p(b)$.

   Therefore, v_p is completely additive and hence, is also additive.

   (b) *Show that for any positive integers a and b,*

$$v_p(a + b) \geqslant \min(v_p(a), v_p(b))$$

   Suppose that $v_p(a) = k$ and $v_p(b) = \ell$. Write $a = p^k m$ for some $m \in \mathbb{Z}$ with $(p, m) = 1$ and write $b = p^\ell n$ for some $n \in \mathbb{Z}$ with $(n, p) = 1$. Without loss of generality, we may assume that $k \geqslant \ell$, so $\min(v_p(a), v_p(b)) = \ell$. Then note that

$$a + b = p^k m + p^\ell n = p^\ell (p^{k-\ell} m + n)$$

   In particular, $p^\ell \mid a + b$, so $\ell \in \{k \in \mathbb{N} : p^k \mid a + b\}$. Therefore

$$\min(v_p(a), v_p(b)) = \ell \leqslant \max\{k \in \mathbb{N} : p^k \mid a + b\} = v_p(a + b)$$

3. *Find all positive integers n with $\sigma(n) = 12$.*

Note that $\sigma(1) = 1$, so we may assume that $n > 1$. Hence, we can write $n = p_1^{e_1} \cdots p_g^{e_g}$ where $p_1, \dots, p_g$ are distinct primes and $e_1, \dots, e_g > 0$. Then using the fact that σ is multiplicative, we find that

$$12 = \sigma(p_1^{e_1} \cdots p_g^{e_g}) = \sigma(p_1^{e_1}) \cdots \sigma(p_g^{e_g})$$

We can conclude that for each prime power divisor of n, we must have $\sigma(p_i^{e_i}) \mid 12$. Therefore, we can first solve $\sigma(p^e) = 1, 2, 3, 4, 6, 12$ for a prime p and exponent $e > 0$ before compiling those solutions into a solution of $\sigma(n) = 12$.

Suppose that $\sigma(p^e) = 1$. Since $1 = \sigma(p^e) \geqslant p^e + 1 > p^e$, we find that $p^e < 1$, a contradiction. Therefore, no prime power has $\sigma(p^e) = 1$.

Suppose that $\sigma(p^e) = 2$. Again, $2 = \sigma(p^e) > p^e$, which is a contradiction. So no prime power has $\sigma(p^e) = 2$.

Suppose that $\sigma(p^e) = 3$. Then $3 = \sigma(p^e) > p^e$ forcing $p^e = 2$, so $p = 2$ and $e = 1$. It is easy to check that $\sigma(2) = 3$.

Suppose that $\sigma(p^e) = 4$. Then $4 = \sigma(p^e) > p^e$ forcing $p = 2$ or $p = 3$. If $p = 2$, then $e > 1$ (else $\sigma(p^e) = 3$ as in the previous case). But $\sigma(2^e) \geqslant \sigma(2^2) = 7 > 4$, so $p \neq 2$. Hence $p = 3$ forcing $e = 1$. Again, one can check that $\sigma(3) = 4$.

We claim that we do not need to check the case when $\sigma(p^e) = 6$. If $\sigma(p_i^{e_i}) = 6$ for some i, then we would also have to have $\sigma(p_j^{e_j}) = 2$ for some j. But we already found that this cannot happen.

Finally, suppose that $\sigma(p^e) = 12$. Then $12 > p^e$. The following table lists the sum of divisors for all prime powers less than 12 (skipping 2 and 3 since we computed their sums of divisors in previous parts):

p^e	4	5	7	8	9	11
$\sigma(p^e)$	7	6	8	15	13	12

and we conclude that the only possibility is $p^e = 11$.

From the above information, we conclude that the only possible values of n with $\sigma(n) = 12$ are $n = 6$ and $n = 11$.

4. *A positive integer $n > 1$ is highly composite if $\tau(m) < \tau(n)$ whenever $m < n$.*

   (a) *Find the first five highly composite numbers*

   Here is a table of values of $\tau(n)$:

n	1	2	3	4	5	6	7	8	9	10	11	12
$\tau(n)$	1	2	2	3	2	4	2	4	3	4	2	6

n	13	14	15	16	17	18	19	20	21	22	23	24
$\tau(n)$	2	4	4	5	2	4	2	5	4	4	2	8

   and so we see that the first five highly composite numbers are 2, 4, 6, 16, and 24.

   (b) *Show that if n is highly composite and m is a positive integer with $\tau(m) > \tau(n)$, then there exists a highly composite integer k so that $n < k \leqslant m$. Conclude that there are infinitely many highly composite integers.*

   Suppose that n is highly composite and m is a positive integer with $\tau(m) > \tau(n)$. There are (at least) two approaches one could take to proving that there exists a highly composite integer between n and m.

Approach 1:

Let $S = \{k : \tau(k) > \tau(n)\}$. Since $\tau(m) > \tau(n)$, we see that $m \in S$ so that $S \neq \varnothing$. By the well-ordering principle, S has a least element, say ℓ. We claim that ℓ is highly composite. To see this, suppose that $k < \ell$. Then $k \notin S$ because ℓ is the least element of S. Hence, $\tau(k) \leqslant \tau(n) < \tau(\ell)$. Therefore, ℓ is highly composite. Moreover, $\ell \leqslant m$ because $m \in S$ and ℓ is the least element of S. Finally, $\ell > n$ because any $k \leqslant n$ has $\tau(k) \leqslant \tau(n)$, so $k \notin S$. Therefore, there is a highly composite integer ℓ with $n < \ell \leqslant m$.

Approach 2:

Notice that $S := \{\tau(k) : n < k \leqslant m\}$ is a finite set and hence, has a maximum. Let $M = \max_{k \in S} \tau(k)$ and suppose that $k \in S$ is the minimal element of S with $\tau(k) = M$ (here, we again use the fact that S is finite). We claim that k is highly composite. To see this, suppose that $\ell < k$. Then if $\ell > n$, we have $\ell \in S$ so $\tau(\ell) \leqslant M = \tau(k)$. But k is the smallest member of S with $\tau(k) = M$, so we must have $\tau(\ell) < M = \tau(k)$. If $\ell \leqslant n$, then since n is highly composite, $\tau(\ell) \leqslant \tau(n) < \tau(m) \leqslant M = \tau(k)$. Therefore, $\tau(\ell) < \tau(k)$, so k is highly composite and $n < k \leqslant m$.

To conclude the proof, we note that the set $\{\tau(n) : n \in \mathbb{N}\}$ is unbounded. In particular, for any prime p and $e \geqslant 0$, $\tau(p^e) = e + 1$ which goes to infinity as e goes to infinity. If, by contradiction, there were only finitely many highly composite integers, let n be the maximal highly composite integer. Since τ is unbounded, there exists $m > n$ so that $\tau(m) > \tau(n)$. By the first part of this problem, there exists $k > n$ so that k is highly composite, contradicting the fact that n is the largest highly composite integer. Hence, there must be infinitely many highly composite integers.