1. Let $n \geqslant 1$. Compute $\varphi(2^n)$ (and make sure to prove that your answer is correct).

Approach 1.

Of the elements in the set $\mathbb{Z}/2^n\mathbb{Z} = \{0, 1, 2, \ldots, 2^n - 1\}$, half are even and half are odd. Since an integer is relatively prime to 2^n if and only if it is odd, we must have that $(\mathbb{Z}/2^n\mathbb{Z})^\times$ is half the size of $\mathbb{Z}/2^n\mathbb{Z}$. Since $\varphi(2^n) = \#(\mathbb{Z}/2^n\mathbb{Z})^\times$, we must have

$$\varphi(2^n) = \frac{\#(\mathbb{Z}/2^n\mathbb{Z})}{2} = \frac{2^n}{2} = 2^{n-1}$$

Approach 2.

We first claim that

$$\left(\mathbb{Z}\middle/2^n\mathbb{Z}\right)^\times = \left\{2k+1 : k \in \mathbb{N}, \quad k < \frac{2^n - 1}{2}\right\}$$

Suppose $x \in (\mathbb{Z}/2^n\mathbb{Z})^\times$. Then x must be relatively prime to 2^n and hence, odd. Moreover, $0 \leqslant x < 2^n$, so writing $x = 2k+1$ for some $k \in \mathbb{N}$ gives $2k + 1 < 2^n$, implying that $k < \frac{2^n-1}{2}$. Therefore $x \in \left\{2k+1 : k \in \mathbb{N}, \quad k < \frac{2^n-1}{2}\right\}$ and so we conclude that

$$\left(\mathbb{Z}\middle/2^n\mathbb{Z}\right)^\times \subseteq \left\{2k+1 : k \in \mathbb{N}, \quad k < \frac{2^n - 1}{2}\right\}$$

For the reverse containment, suppose that $x \in \left\{2k+1 : k \in \mathbb{N}, \quad k < \frac{2^n-1}{2}\right\}$. Then x is relatively prime to 2^n since x is odd and moreover, writing $x = 2k+1$ with $0 \leqslant k < \frac{2^n-1}{2}$ gives $0 \leqslant 2k+1 = x$ and $x = 2k+1 < 2^n$. Hence, $x \in (\mathbb{Z}/2^n\mathbb{Z})^\times$. Therefore,

$$\left(\mathbb{Z}\middle/2^n\mathbb{Z}\right)^\times \supseteq \left\{2k+1 : k \in \mathbb{N}, \quad k < \frac{2^n - 1}{2}\right\}$$

Hence, we conclude that

$$\left(\mathbb{Z}\middle/2^n\mathbb{Z}\right)^\times = \left\{2k+1 : k \in \mathbb{N}, \quad k < \frac{2^n - 1}{2}\right\}$$

Since $\varphi(2^n) = \#(\mathbb{Z}/2^n\mathbb{Z})^\times$, we must have

$$\varphi(2^n) = \lfloor\frac{2^n - 1}{2}\rfloor + 1 = 2^{n-1}$$

2. Use Euler's theorem to find the last digit of the decimal expansion of 13^{999999}.

   Note that
$$\left(\mathbb{Z}/10\mathbb{Z}\right)^{\times} = \{1, 3, 7, 9\}$$
   so $\varphi(10) = 4$. Since $(13, 10) = 1$, Euler's theorem implies that $13^4 \equiv 1 \mod 10$. Therefore,
$$13^{999999} \equiv 13^{4\cdot 249999+3} \equiv (13^4)^{249999} \cdot 13^3 \equiv 3^3 \equiv 7 \mod 10$$
   Therefore, the last digit of 13^{999999} is 7.

3. Find the last digit of the decimal expansion of 2^{999999}.

   Approach 1.

   We first claim that $2^{a+4k} \equiv 2^a \mod 10$ whenever $k \geqslant 0$ and $a \geqslant 1$. To prove this, observe that $2^{a+4k} - 2^a = 2^a(2^{4k} - 1) = 2^a(16^k - 1)$. Since $16 \equiv 1 \mod 5$, we must have $16^k \equiv 1 \mod 5$ and hence, $16^k - 1$ is divisible by 5. Moreover, 2^a is divisible by 2 since $a \geqslant 1$. Therefore, $2^a(16^k - 1) = 2^{a+4k} - 2^a$ is divisible by 10.

   Using the fact we just proved with $k = 249999$ and $a = 3$, we have
$$2^{999999} = 2^{3+4\cdot 249999} \equiv 2^3 \equiv 8 \mod 10$$
   Therefore, the last digit of 2^{999999} is 8.

   Approach 2.

   First observe that $2^{999999} \equiv 0 \mod 2$. Additionally since $(2, 5) = 1$, Fermat's Little Theorem implies that
$$2^{999999} = 2^{3+4\cdot 249999} \equiv 2^3 \cdot (2^4)^{249999} \equiv 8 \equiv 3 \mod 5$$
   By Sun-Tsu's Theorem, the only numbers which are equivalent to $0 \mod 2$ and $3 \mod 5$ are equivalent to $8 \mod 10$, so $2^{999999} \equiv 8 \mod 10$. Hence, the last digit of 2^{999999} is 8.