1. Which positive integers have an odd number of positive divisors?

   Since 1 has 1 positive divisor, 1 has an odd number of positive divisors. Now suppose that $n > 1$ has an odd number of positive divisors. Factoring $n = p_1^{e_1} \cdots p_g^{e_g}$ for distinct primes $p_1, \ldots, p_g$ and $e_1, \ldots, e_g \geqslant 1$ yields that

   $$\tau(n) = (e_1 + 1) \cdots (e_g + 1)$$

   is odd. But $(e_1 + 1) \cdots (e_g + 1)$ is odd if and only if each e_i is even. But this is equivalent to stating that n is a square. Hence, the set of positive integers with an odd number of positive divisors is equal to the set of perfect squares.

2. What is the product of all positive divisors of a positive integer n?

   Case 1: n is not a perfect square.

   By the previous problem, n has an even number of positive divisors. In fact, those divisors come in pairs of the form $(d, \frac{n}{d})$ for each positive divisor $d < \sqrt{n}$. There are $\tau(n)/2$ such pairs and the product of two elements in a pair is n. Hence, the product of all positive divisors of n is $n^{\tau(n)/2}$.

   Case 2: n is a perfect square.

   This time there are an odd number of positive divisors. All but one divisor come in pairs of the form $(d, \frac{n}{d})$. There are $\frac{\tau(n)-1}{2}$ such pairs with $d < \sqrt{n}$ since the divisor $\sqrt{n}$ does not get included in any such pair. The product of all paired divisors is then $n^{(\tau(n)-1)/2}$ and multiplying by the remaining factor of $n^{1/2}$ gives that the product of all positive divisors of n is

   $$n^{\frac{\tau(n)-1)}{2}+\frac{1}{2}} = n^{\tau(n)/2}$$

   Therefore, in either case the product of all positive divisors of n is equal to $n^{\tau(n)/2}$

3. Define the Möbius function, $\mu : \mathbb{Z}_{>0} \to \mathbb{N}$ so that

$$\mu(n) = \begin{cases} 1 & n = 1 \\ (-1)^r & n = p_1 \cdots p_r \text{ where } p_1, \ldots, p_r \text{ are distinct primes} \\ 0 & \text{otherwise} \end{cases}$$

Show that $\mu(n)$ is multiplicative.

Suppose that m and n are relatively prime. If $m = 1$, then $\mu(mn) = \mu(n) = \mu(m)\mu(n)$ and so we are done. Else, if there exists a prime p so that $p^2 \mid m$, then $p^2 \mid mn$ and so we have $\mu(mn) = 0 = \mu(m)\mu(n)$. By symmetry, if $n = 1$ or if n is not squarefree, then $\mu(mn) = \mu(m)\mu(n)$.

The final case to consider is when m and n are products of distinct primes. Write $m = p_1 \cdots p_g$ and $n = q_1 \cdots q_s$. Note that we never have $p_i = q_j$ since m and n are relatively prime. Therefore mn is the product of distinct primes as well and

$$\mu(mn) = \mu(p_1 \cdots p_g q_1 \cdots q_s) = (-1)^{g+s} = \mu(m)\mu(n)$$

Therefore, μ is multiplicative.

4. Compute $p(6)$, $p^D(6)$, and $p_O(6)$ where O is the set of positive odd integers.

   The partitions of 6 are as follows:

$$\begin{aligned}
6 &= 6 \\
&= 5+1 \\
&= 4+2 \\
&= 4+1+1 \\
&= 3+3 \\
&= 3+2+1 \\
&= 3+1+1+1 \\
&= 2+2+2 \\
&= 2+2+1+1 \\
&= 2+1+1+1+1 \\
&= 1+1+1+1+1+1
\end{aligned}$$

   We can immediately see that there are 11 partitions of 6, so $p(6) = 11$. The partitions of 6 into distinct parts are

$$\begin{aligned}
6 &= 6 \\
&= 5+1 \\
&= 4+2 \\
&= 3+2+1
\end{aligned}$$

   so $p^D(6) = 4$. The partitions of 6 into odd parts are

$$\begin{aligned}
6 &= 5+1 \\
&= 3+3 \\
&= 3+1+1+1 \\
&= 1+1+1+1+1+1
\end{aligned}$$

   and so $p_O(6) = 4$ as well.

5. Show that for all $n \geqslant 1$,
$$p(n) = p(n-1) + p_2(n)$$

Let S_n be the set of all partitions of n. Note that for any $(\lambda_1, \ldots, \lambda_r) \in S_{n-1}$, we have that $(\lambda_1, \ldots, \lambda_r, 1) \in S_n$. Hence, we can define a function

$$\begin{aligned} f : S_{n-1} &\to S_n \\ (\lambda_1, \ldots, \lambda_r) &\mapsto (\lambda_1, \ldots, \lambda_r, 1) \end{aligned}$$

Note that f is injective because if $f(\lambda_1, \ldots, \lambda_r) = f(\rho_1, \ldots, \rho_s)$, then we have $(\lambda_1, \ldots, \lambda_r, 1) = (\rho_1, \ldots, \rho_s, 1)$ and so $r = s$ and $(\lambda_1, \ldots, \lambda_r) = (\rho_1, \ldots, \rho_s)$. Moreover, the image of f is exactly the set of partitions of n which have 1 as a part. This is the complement of the set of partitions which have all parts $\geqslant 2$, so we find that S_n decomposes as the disjoint union of the image of f with the set of partitions which have all parts $\geqslant 2$. As a consequence, we now have that $p(n) = p(n-1) + p_2(n)$.