1. Find all positive integers n with $\varphi(n) = 4$.

Since $\varphi(1) = 1$, any integer n with $\varphi(n) = 4$ must have $n > 2$ and hence, have a prime factorization. Suppose $n = p_1^{e_1} \cdots p_g^{e_g}$ for distinct primes $p_1, \ldots, p_g$ and $e_1, \ldots, e_g \geqslant 1$. Then we have

$$4 = \varphi(n) = p_1^{e_1-1} \cdots p_g^{e_g-1}(p_1 - 1) \cdots (p_g - 1)$$

Suppose first that n has some odd prime divisor, p. Then $p - 1 \mid 4$, so the only possibilities for p are 3 and 5. Moreover, if either 3^2 or 5^2 divides n, then $3 \mid 4$ or $5 \mid 4$ which is a contradiction, so 3 and 5 divide n at most once. If $5 \mid n$, then

$$4 = \varphi(n) = \varphi(5)\varphi(2^{e_1} \cdot 3^{e_2}) = 4\varphi(2^{e_1} \cdot 3^{e_2})$$

so $1 = \varphi(2^{e_1} \cdot 3^{e_2})$. The only integers k with $\varphi(k) = 1$ are $k = 1$ and $k = 2$, so the only n with $\varphi(n) = 4$ and $5 \mid n$ are $n = 5$ and $n = 10$.

Now suppose that $5 \nmid n$. If $3 \mid n$, then we have

$$4 = \varphi(3) \cdot \varphi(2^{e_1}) = 2 \cdot 2^{e_1-1} = 2^{e_1}$$

so we must have $e_1 = 2$. In particular, this implies that $n = 12$.

Finally, suppose that n is a power of 2. Since $4 = \varphi(n) = \varphi(2^{e_1}) = 2^{e_1-1}$, we must have $e_1 = 3$ and so $n = 8$.

In conclusion, the only positive integers n with $\varphi(n) = 4$ are $n = 5, 8, 10$, and 12.

2. Do there exist $m, n \in \mathbb{Z}_{>0}$ so that $(m, n) > 1$ and $\varphi(mn) = \varphi(m)\varphi(n)$? What does this allow you to say about the truth of the claim "$(m, n) = 1$ if and only if $\varphi(mn) = \varphi(m)\varphi(n)$?"

First define M to be the set of primes dividing m but not n, define N to be the set of primes dividing n but not m, and define B to be the set of primes dividing both. Let

$$\mathcal{M} = \prod_{p \in M} \left(1 - \frac{1}{p}\right)$$
$$\mathcal{N} = \prod_{p \in N} \left(1 - \frac{1}{p}\right)$$
$$\mathcal{B} = \prod_{p \in B} \left(1 - \frac{1}{p}\right)$$

Then observe that

$$\varphi(mn) = mn\mathcal{M}\mathcal{N}\mathcal{B}$$

and on the other hand

$$\varphi(m)\varphi(n) = mn\mathcal{M}\mathcal{N}$$

The factor that appears in the former calculation but not the latter is $\mathcal{B}$ which we can express using the fact that

$$\varphi((m, n)) = (m, n)\mathcal{B}$$

Hence, we find that

$$\varphi(mn) = mn\mathcal{M}\mathcal{N}\mathcal{B} = \varphi(m)\varphi(n)\mathcal{B} = \frac{\varphi(m)\varphi(n)\varphi((m, n))}{(m, n)}$$

As a consequence, the only way that we can have $\varphi(m)\varphi(n) = \varphi(mn)$ is if $\varphi((m, n)) = (m, n)$. But the only integer k with $\varphi(k) = k$ is $k = 1$, so the only way $\varphi(m)\varphi(n) = \varphi(mn)$ is if $(m, n) = 1$. The provides the converse to the multiplicativity of φ, so we find that $\varphi(mn) = \varphi(m)\varphi(n)$ if and only if $(m, n) = 1$.

3. For which positive integers n does $\varphi(3n) = 3\varphi(n)$?

   We first note that n must be divisible by 3. If not, then we would have $3\varphi(n) = \varphi(3n) = \varphi(3)\varphi(n) = 2\varphi(n)$, which is a contradiction.

   Next, we claim that if $3 \mid n$, then $\varphi(3n) = 3\varphi(n)$. To see this, write $n = 3^e \cdot f$ where $e \geqslant 1$ and $3 \nmid f$. Now we have

$$\varphi(3n) = \varphi(3^{e+1} f) = 2 \cdot 3^e \cdot \varphi(f) = 3 \cdot 2 \cdot 3^{e-1} \cdot \varphi(f) = 3 \cdot \varphi(3^e) \cdot \varphi(f) = 3\varphi(n)$$

4. Let $n \in \mathbb{Z}_{>0}$. Show that there exists $r \geqslant 1$ so that $\varphi^r(n) = 1$. Here, $\varphi^r(n)$ means the rth iterate of φ, i.e. $\varphi(\varphi(\varphi(\cdots\varphi(n))))$ where φ is composed with itself r times.

   Let $S = \{\varphi^r(n) : r \geqslant 1\}$. Since S is a nonempty subset of positive integers, it has a least element. Let k be the minimal element of S and suppose by contradiction that $k > 1$. Since $k \in S$, $\varphi(k) \in S$ and since $k > 1$, k has a prime factor, so $\varphi(k) = k \cdot \prod_{p|k}\left(1 - \frac{1}{p}\right) < k$. But then $\varphi(k) \in S$ and is smaller than the minimal element of S, a contradiction. Therefore, the minimum of S must be 1. As a result, there exists $r \geqslant 1$ so that $\varphi^r(n) = 1$.