1. *For a positive integer n, show that*

$$\tau(n^2) = \#\{(a,b) : a,b \in \mathbb{Z}_{>0}, \operatorname{lcm}(a,b) = n\}$$

Case 1: $n = 1$

In this case, $n^2 = 1$, so $\tau(n^2) = \tau(1) = 1$. Moreover, if $a, b \in \mathbb{Z}_{>0}$ and $\operatorname{lcm}(a,b) = 1$, then $a, b \mid 1$ and so we must have $a = b = 1$. Hence,

$$\#\{(a,b) : a,b \in \mathbb{Z}_{>0}, \operatorname{lcm}(a,b) = 1\} = \#\{(1,1)\} = 1 = \tau(n^2)$$

which completes the first case.

Case 2: $n > 1$

First write $n = p_1^{e_1} \cdots p_g^{e_g}$ for distinct primes $p_1, \ldots, p_g$ and $e_1, \ldots, e_g \geqslant 1$. Hence

$$\tau(n^2) = \tau(p_1^{2e_1})\tau(p_2^{2e_2}) \cdots \tau(p_g^{2e_g}) = (2e_1 + 1)(2e_2 + 1) \cdots (2e_g + 1)$$

Next, suppose that $a, b \in \mathbb{Z}_{>0}$ with $\operatorname{lcm}(a,b) = 1$. Then $a, b \mid n$ and so we can write $a = p_1^{a_1} \cdots p_g^{a_g}$ and $b = p_1^{b_1} \cdots p_g^{b_g}$ where $a_1, \ldots, a_g, b_1, \ldots, b_g \geqslant 0$. Moreover, we know that

$$p_1^{e_1} \cdots p_g^{e_g} = n = \operatorname{lcm}(a,b) = p_1^{\max(a_1,b_1)} \cdots p_g^{\max(a_g,b_g)}$$

As a consequence, we must have

$$e_i = \max(a_i, b_i)$$

for each $1 \leqslant i \leqslant g$.

If a and b are positive integers of the form $a = p_1^{a_1} \cdots p_g^{a_g}$ and $b = p_1^{b_1} \cdots p_g^{b_g}$ where

$$a_1, \ldots, a_g, b_1, \ldots, b_g \geqslant 0 \text{ and } e_i = \max(a_i, b_i) \text{ for all } 1 \leqslant i \leqslant g$$

then $\operatorname{lcm}(a,b) = 1$. As a result, we conclude that the number of pairs (a,b) where a and b are positive integers with $\operatorname{lcm}(a,b) = 1$ is equal to the number of pairs $((a_1, \ldots, a_g), (b_1, \ldots, b_g))$ where $a_1, \ldots, a_g, b_1, \ldots, b_g \geqslant 0$ and $e_i = \max(a_i, b_i)$ for each $1 \leqslant i \leqslant g$.

Note that for any $1 \leqslant i \leqslant g$, there are $2e_i + 1$ choices for a_i and b_i: if $a_i = e_i$, then b_i can be any of the $e_i + 1$ values $0, 1, 2, \ldots, e_i$. Otherwise, $b_i = e_i$ and a_i can be any of the $e_i + 1$ values $0, 1, 2, \ldots, e_i$. Hence we have $2(e_i + 1) - 1 = 2e_i + 1$ choices for a_i and b_i because we have counted the possibility $a_i = b_i = e_i$ twice.

Hence, there are $(2e_1 + 1)(2e_2 + 1) \cdots (2e_g + 1) = \tau(n^2)$ pairs (a,b) satisfying $\operatorname{lcm}(a,b) = n$.

2. *A partition of n is said to be self-conjugate if it is its own conjugate.*

   (a) *Suppose that $(\lambda_1, \dots, \lambda_r)$ is a partition of n. Let $S = \{\lambda_j : \lambda_j \geqslant j\}$. Show that if $\lambda_k \in S$, then $\lambda_{k-1} \in S$. Conclude that S has the form $\{\lambda_1, \dots, \lambda_t\}$ for some t.*

   Suppose that $\lambda_k \in S$. Then we find that

$$\lambda_{k-1} \geqslant \lambda_k \geqslant k > k-1$$

   and so $\lambda_{k-1} \in S$. As a consequence, if we let $t = \max\{j : \lambda_j \geqslant j\}$, we see that $\lambda_1, \dots, \lambda_t \in S$ but for any $i > t$, $\lambda_i \notin S$. Hence, $S = \{\lambda_1, \dots, \lambda_t\}$.

   (b) *Let $(\lambda_1, \dots, \lambda_r)$ be a self-conjugate partition of n. Suppose that a dot in the Ferrers diagram of $(\lambda_1, \dots, \lambda_r)$ is in row j and column ℓ. Show that either $\lambda_j \geqslant j$ or $\lambda_\ell \geqslant \ell$.*

   If $\lambda_j \geqslant j$, we are done. Otherwise, $\lambda_j < j$. Since λ_j gives the length of row j in the Ferrers diagram, the column index of our dot, ℓ, must be less than or equal to the length of the row, λ_j. So now we have that $\ell \leqslant \lambda_j < j$.

   Since $(\lambda_1, \dots, \lambda_r)$ is self-conjugate, there exists a dot in row ℓ and column j. Again, the column index, j, must be less than or equal to the length of the row, λ_ℓ. So now we have $\ell < j \leqslant \lambda_\ell$, which is what we needed to show.

(c) *Let $(\lambda_1, \ldots, \lambda_r)$ be a self-conjugate partition of n. Let $k = \#\{j : \lambda_j \geqslant j\}$. For each $1 \leqslant i \leqslant k$, define $\rho_i = 2\lambda_i - (2i-1)$. Prove that $(\rho_1, \ldots, \rho_k)$ is a partition of n with distinct, odd parts.*

We first show that $\rho_1, \ldots, \rho_k$ is a partition with distinct, odd parts. First note that all of the ρ_i are positive because the set S from part (a) has the form $\{\lambda_1, \ldots, \lambda_k\}$ and so for each $1 \leqslant i \leqslant k$, we have

$$\rho_i = 2\lambda_i - (2i-1) \geqslant 2i - 2i + 1 > 0$$

Next, we claim that $\rho_1 > \rho_2 > \cdots > \rho_k$. Pick some i with $1 \leqslant i < k$. Then we note that because $\lambda_i \geqslant \lambda_{i+1}$, we have $2\lambda_i \geqslant \lambda_{i+1}$. Moreover, $2i-1 < 2i+1$, so $-(2i-1) > -(2i+1)$ and combining inequalities yields

$$\rho_{i+1} = 2\lambda_{i+1} - (2i+1) < 2\lambda_i - (2i-1) = \rho_i$$

To see that each ρ_i is odd, observe that $\rho_i = 2\lambda_i - (2i-1)$ is the difference between an even number and an odd number, so it must be odd.

Hence, $(\rho_1, \ldots, \rho_k)$ is a partition with distinct, odd, parts. The only question that remains is whether $(\rho_1, \ldots, \rho_k)$ is a partition of n. To accomplish this goal, we show that ρ_i counts the dots in a particular subset of the Ferrers diagram for $(\lambda_1, \ldots, \lambda_r)$. Then we show that no dot is counted by multiple ρ_i and that each dot is counted by some ρ_i. In particular, each ρ_i counts the number of dots in the Ferrers diagram for $(\lambda_1, \ldots, \lambda_r)$ which live in either row i or column i, but do not live in any row or column with index less than i. I.e. ρ_i counts the number of dots in the Ferrers diagram which live below and to the right of the dot in column i and row i.

Suppose that some dot in the Ferrers diagram for $(\lambda_1, \ldots, \lambda_r)$ were counted by ρ_i. Then that dot does not live in a row or column with index less than i, so it cannot be counted by any ρ_j with $j < i$. Moreover, that dot cannot be counted by a ρ_j with $i < j$ because that dot lives in a row or column with index less than j. Therefore, $\rho_1, \ldots, \rho_k$ do not double count any dots in the Ferrers diagram for $\lambda_1, \ldots, \lambda_r$. As a result, we conclude that

$$\sum_{j=1}^{k} \rho_j \leqslant n$$

To see that the sum is equal to n, we need to show that every dot in the Ferrers diagram for $(\lambda_1, \ldots, \lambda_r)$ is counted by some ρ_i. By part (b), every dot either lives in a row j with λ_j or a column ℓ with $\lambda_\ell \geqslant \ell$. But then that dot will be counted by either ρ_j or ρ_ℓ. Hence, every dot is counted by some ρ_i.

Now that we have shown that every dot is counted by some ρ_i and that no dot is counted twice, we can conclude that

$$\sum_{j=1}^{k} \rho_j = n$$

and hence, $(\rho_1, \ldots, \rho_k)$ is a partition of n into distinct, odd parts.

(d) *Let O be the set of odd positive integers. Show that the number of self-conjugate partitions of n is equal to $p_O^D(n)$.*

Let S be the set of self-conjugate partitions of n and let T be the set of partitions of n into distinct, odd parts. Part (c) informs us that there is a well-defined function

$$\begin{aligned} f : S &\to T \\ (\lambda_1, \ldots, \lambda_r) &\mapsto (\rho_1, \ldots, \rho_k) \end{aligned}$$

Moreover, we can define a function $g : T \to S$ as follows. Given a partition of n into distinct, odd parts $(\rho_1, \ldots, \rho_k)$, we can define a Ferrers diagram for a new partition of n using the following procedure. For each $1 \leqslant i \leqslant k$, place one dot, D_i in row i, column i and then place $\frac{\rho_i - 1}{2}$ dots to the right of D_i and $\frac{\rho_i - 1}{2}$ dots below D_i. Since $\rho_1 > \rho_2 > \cdots > \rho_k$, this indeed yields a Ferrers diagram for a partition of n. Lets say that partition is $(\lambda_1, \ldots, \lambda_r)$. Since column j in the Ferrers diagram for $(\lambda_1, \ldots, \lambda_r)$ will have the same number of dots as row j of the same Ferrers diagram, the Ferrers diagram for $(\lambda_1, \ldots, \lambda_r)$ is self-conjugate. As a consequence, we can define

$$\begin{aligned} g : T &\to S \\ (\rho_1, \ldots, \rho_k) &\mapsto (\lambda_1, \ldots, \lambda_r) \end{aligned}$$

f and g are inverses and since S and T are finite sets, we conclude that $|S| = |T|$.

3. *Use the result from the previous problem to show that $p(n)$ is odd if and only if $p_O^D(n)$ is odd.*

Let S be the set of all partitions of n. Let C be the set of self-conjugate partitions of n and let $N = S \setminus C$ be the set of all non-self-conjugate partitions of n. Let $c : S \to S$ be the conjugation map. Since any partition $(\lambda_1, \dots, \lambda_r)$ which is not self-conjugate has $(\lambda'_1, \dots, \lambda'_{\lambda_1})$ also not self-conjugate and since conjugation is its own inverse, we see that we can pair up elements of N by pairing $(\lambda_1, \dots, \lambda_r)$ with its conjugate $(\lambda'_1, \dots, \lambda'_{\lambda_1})$. Since the elements of N can be evenly paired, the size of N must be even. Therefore, we see that

$$|S| = |N| + |C| \equiv |C| \mod 2$$

and so $p(n) = |S|$ is odd if and only if $|C|$ is odd. By problem 3 however, $|C| = p_O^D(n)$.

4. *(Extra Credit) Use Ferrers diagrams to show that $p^D(n)$ is equal to the number of partitions of the form $(\lambda_1, \dots, \lambda_k)$ where for each i so that $1 \leqslant i \leqslant \lambda_1$, there exists a j so that $i = \lambda_j$.*

Let

$D = \{(\lambda_1, \dots, \lambda_k) : (\lambda_1, \dots, \lambda_k)$ is a partition of n with distinct parts$\}$
$E = \{(\lambda_1, \dots, \lambda_k) : (\lambda_1, \dots, \lambda_k)$ is a partition of n and for each $1 \leqslant i \leqslant \lambda_1$ there exists j so that $\lambda_j = i\}$

We first claim that for any $(\lambda_1, \dots, \lambda_k) \in E$, the conjugate partition of $(\lambda_1, \dots, \lambda_k)$ is in D. Let $(\lambda'_1, \dots, \lambda'_{\lambda_1})$ be the conjugate partition of $(\lambda_1, \dots, \lambda_k)$ and suppose by contradiction that there exists an i so that $\lambda'_i = \lambda'_{i+1}$. Then

$$\#\{\lambda_j : \lambda_j \geqslant i+1\} = \lambda'_{i+1} = \lambda'_i = \#\{\lambda_j : \lambda_j \geqslant i\}$$

and since $\{\lambda_j : \lambda_j \geqslant i+1\} \subseteq \{\lambda_j : \lambda_j \geqslant i\}$, we conclude that

$$\{\lambda_j : \lambda_j \geqslant i+1\} = \{\lambda_j : \lambda_j \geqslant i\}$$

But since $(\lambda_1, \dots, \lambda_k) \in E$, there exists k so that $\lambda_k = i$ and we find that $\lambda_k \notin \{\lambda_j : \lambda_j \geqslant i+1\}$ but $\lambda_k \in \{\lambda_j : \lambda_j \geqslant i\}$. Therefore,

$$\{\lambda_j : \lambda_j \geqslant i+1\} \subsetneq \{\lambda_j : \lambda_j \geqslant i\}$$

which is a contradiction. Hence, we must have $\lambda'_1 > \lambda'_2 > \cdots > \lambda'_{\lambda_1}$ and so the conjugate partition of $(\lambda_1, \dots, \lambda_k)$ is in D.

This indicates that the conjugate map $E \to D$ is well-defined and since the conjugate map is invertible, it must be injective.

Moreover, we claim that the conjugate of any partition in D is in E. Suppose that $(\lambda_1, \dots, \lambda_k)$ is a partition of n with distinct parts and let $(\lambda'_1, \dots, \lambda'_{\lambda_1})$ be its conjugate partition. Pick an i so that $1 \leqslant i \leqslant \lambda'_1 = k$. Because $\lambda_1 > \lambda_2 > \cdots > \lambda_k$, we know that

$$\lambda'_{\lambda_i} = \#\{\lambda_j : \lambda_j \geqslant \lambda_i\} = \#\{\lambda_1, \dots, \lambda_i\} = i$$

Hence, $(\lambda'_1, \dots, \lambda'_{\lambda_1}) \in E$.

Therefore, conjugation defines an injective map $D \to E$. Since E and D are finite sets, we conclude that $\#E = \#D$ which is what we wanted to show.