Section 1.3 – Induction I

Axiom: Suppose $S \subseteq \mathbb{N}$, $0 \in S$, and if $k \in S$, then $k+1 \in S$. Then $S = \mathbb{N}$

↳ called "The Principle of Mathematical Induction"

Q How does this relate to proof by induction?

A:

Principle of Ind.	Process of proof by ind.
start with $S = \{ n \in \mathbb{N} \mid \varphi(n) \text{ is true} \}$	Start with: some property we want to prove, $\varphi(n)$ → statement about n: e.g. $\varphi(n)$ abbrs "n is even or odd"
① $0 \in S$	① Base case: $\varphi(0)$ is true. \varphi – φ
② $n \in S \longrightarrow n+1 \in S$	② Induction step. Assume $\varphi(n)$ is true, prove $\varphi(n+1)$ is true
③ $S = \mathbb{N}$	③ Conclude: $\forall n\ \varphi(n)$ is true. "for all"

Equivalent Axioms

Claim: The Well-Ordering Principle is equivalent to the principle of induction

Partial proof (well-ordering $\Rightarrow$ induction).

Assume the well-ordering principle holds.

Suppose $S \subseteq \mathbb{N}$, $0 \in S$, and that for each $k \in S$, $k+1 \in S$.

By ↯, assume $S \neq \mathbb{N}$

Then $X = \mathbb{N} \setminus S$ is nonempty

By well-ordering, X has a least elt, x.

Since x is the smallest elt of X, $x-1 \boxed{\notin} X$ not in

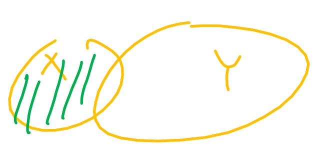
Hence, $x-1 \in S$

So $\underbrace{(x-1)+1}_{= x} \in S$ and $x \in X = \mathbb{N} \setminus S$

Contradiction

Therefore, $S = \mathbb{N}$

Aside: $X \setminus Y = X - Y = \{x \in X \mid x \notin Y\}$



Axiom: The Principle of Strong Induction:

Suppose $S \subseteq \mathbb{N}$, $0 \in S$, and for each k,

if $0, 1, 2, \dots, k \in S$ then $k+1 \in S$.

Then $S = \mathbb{N}$

Fact: Induction $\Rightarrow$ Strong Induction.

"anything you can prove with strong induction, you can prove with induction"

Claim: Strong Induction $\Rightarrow$ Well-Ordering

Proof: ~~Assume Strong Induction~~

~~WTS: Every nonempty $X \subseteq \mathbb{N}$ has a least elt.~~

Suppose $X \subseteq \mathbb{N}$, $X \neq \emptyset$, X has no least elt.

WTS: Strong Induction does not hold

↳ note: contrapositive

Take $S = \mathbb{N} \setminus X$

Note $0 \in S$ (if $0 \in X$, X would have a least elt.)

Suppose $0, 1, \ldots, k \in S$.

Note $k+1 \notin X$ (if $k+1 \in X$, X would have a least elt.)

S satisfies strong ind. hypothesis.

But $S \neq \mathbb{N}$ because $X \neq \emptyset$

S fails the strong ind. conclusion

So strong ind. does not hold.

Examples

① Show that $\forall n: \; n^2 = \sum_{j=1}^{n} (2j-1)$

Recall: $\sum_{j=1}^{n} (2j-1) = (2\cdot 1 - 1) + (2\cdot 2 - 1) + (2\cdot 3 - 1) + \cdots + (2n-1)$

$= 1 + 3 + 5 + \cdots + (2n-1)$

Base case: $n = 1$: LHS: $n^2 = 1^2 = 1$

RHS: $\sum_{j=1}^{1} (2j-1) = 2\cdot 1 - 1 = 1$

Step: Assume $n^2 = \sum_{j=1}^{n} (2j-1)$

WTS: $(n+1)^2 = \sum_{j=1}^{n+1} (2j-1)$

Note: $(n+1)^2 = n^2 + 2n+1$

$= \sum_{j=1}^{n} (2j-1) + 2n+1$

$= \sum_{j=1}^{n} (2j-1) + 2(n+1) - 1$

$= \sum_{j=1}^{n+1} 2j-1$

(2) Show that $\forall n \geq 4: 2^n < n!$

Base case: $n = 4$

LHS: $2^4 = 16$

RHS: $4! = 24$

and $16 < 24$

Step: Assume $\underline{2^n < n!}$

WTS: $2^{n+1} < (n+1)!$

$2^{n+1} = 2^n \cdot 2 < n! \, 2 < n! \cdot \underbrace{(n+1)}_{=(n+1)!}$