Chapter 4: Finite Dimensional Real Vector Spaces

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February 28, 2024

1 Introduction

1.1 Definition

Def: A *vector in* $\mathbb{R}^n$ is a $1 \times n$ or $n \times 1$ matrix with real number entries.

- The notation $\mathbb{R}^n$ refers to the set of every vector with n real numbers.

Def: The *dimension* of $\mathbb{R}^n$ is n.

1.2 Visualizing Vectors

- Let's start with $\mathbb{R}^2$
- We're very familiar with visualizing points of the form (x, y)
- Draw picture of the plane and the point $(2, 1)$.
- Rather than just thinking of vectors as points, however, it can sometimes be easiest to think about them as arrows.
- Draw arrow $(2, 1)$
- In fact, we can draw this arrow anywhere we like and it's still the same arrow

Def: A vector whose base is drawn at the origin is said to be in *standard position*.

- If we have two points, A and B, we denote the vector from A to B as $\vec{AB}$.

1.3 Visualizing Vector Operations

- We have two major vector operations we want to get a picture of: addition and scalar multiplication
- We know how to add two vectors already: $(2, 1) + (3, -2) = (5, -1)$.
- What does this look like geometrically? (picture)
- More generally, when we add vectors, we connect them tip-to-tail.
- Note that the arithmetic version of addition is obviously commutative
- But the geometric version isn't so obvious
- What about scalar multiplication?
- Note $3(2, 1) = (6, 3)$, which takes the vector $(2, 1)$ and scales its length by 3.
- Note $-4(2, 1) = (-8, -2)$, which takes the vector $(2, 1)$, flips its direction, then multiplies its length by 4.

- Addition and multiplication together give us subtraction
- But another way to do it is to think of $\vec{v} - \vec{u}$ as "the thing you add to $\vec{u}$ to get $\vec{v}$."
- In this sense, $\vec{v} - \vec{u}$ is the vector that goes "to $\vec{v}$ from $\vec{u}$."

Ex: Let $\vec{v} = (1, 2)$ and $\vec{u} = (0, -1)$. Find the vector that goes from the midpoint of $\vec{v}$ to the midpoint of the tips of $\vec{v}$ and $\vec{u}$.

1.4 Geometry Informing Arithmetic

- One thing that pictures can tell us that lists of numbers can't is that vectors should have a length!
- What is the length of the vector $(2, 1)$?
- What is the length of the vector (a, b)?
- **Notation:** For a vector $\vec{v} = (a, b)$, the *norm/length/magnitude* of $\vec{v}$ is $\|\vec{v}\| = \sqrt{a^2 + b^2}$.

Ex: What is the distance between the points (x, y) and (a, b)?

1.5 Moving up to Higher Dimensions

- How do we visualize vectors in $\mathbb{R}^3$?
- Draw the xyz axis system: x out of the page, y to the right, z up.
- Draw the points $(1, 2, 3)$ and $(-3, -2, -1)$.
- Addition and scalar multiplication work exactly as they did before.
- Length is a tricker one though.
- The length of $(1, 2, 3)$ is trickier to compute with the Pythagorean theorem.
- Start by observing that $(1, 2, 3) = (1, 0, 0) + (0, 2, 0) + (0, 0, 3)$.
- Pythagorean theorem says that the length of $(1, 0, 0) + (0, 2, 0)$ is $\sqrt{5}$.
- Then again, we get that the length of $(1, 2, 3)$ is $\sqrt{14}$.
- So in general, the length of (a, b, c) is $\sqrt{a^2 + b^2 + c^2}$.
- In $\mathbb{R}^n$, the length of $(x_1, x_2, \ldots, x_n)$ is $\sqrt{x_1^2 + \cdots + x_n^2}$.

Q: How does length interact with scalar multiplication?

- Compute the length of $3(2, 1)$.
- Compute the length of $-4(2, 1)$.
- Generally: $\|c\vec{v}\| = |c| \cdot \|\vec{v}\|$

Q: How does length interact with vector addition?

- It's complicated: $\|\vec{v} + \vec{w}\| \leqslant \|\vec{v}\| + \|\vec{w}\|$, but we can't say more than this.

1.6 Quantity and Direction

- A common definition of vector that you'll see is that "a vector is a quantity with magnitude and direction."
- I think this is a little too vague to be a good definition, which is why I prefer to say that a vector is a list of numbers.
- However, every vector has a magnitude and direction.
- We know how to find magnitude, but what do we mean by direction?
- The easiest way to represent direction is with a vector of length 1.

Def: A *unit vector* is a vector with length 1.

- **Ex:** $(1,0,0),(0,1,0),(0,0,1),(1/\sqrt{3},1/\sqrt{3},-1/\sqrt{3}),(1/\sqrt{2},0,-1/\sqrt{2})$ are all unit vectors.
- **Fact:** If $\vec{v}\neq 0$, then $\frac{1}{\|\vec{v}\|}\vec{v}$ is a unit vector.
- **Ex:** Consider $\vec{v}=(-1,3,2)$. Show that $\frac{1}{\|\vec{v}\|}\vec{v}$ is a unit vector.
  - First compute $\frac{1}{\|\vec{v}\|}\vec{v}$ (and note that it's a vector!)
  - Second, compute the length of this vector.
- **Ex:** Find the vector with length 10 and which points in the same direction as $(1/\sqrt{2},0,-1/2,1/2)$.
- **Ex:** Let $P=(1,-2,1),Q=(-3,0,5),X=(2,-1,5)$, and $Y=(4,-2,3)$. Is $\vec{PQ}$ parallel to $\vec{XY}$?
  - Compute the unit vectors for $\vec{PQ}$ and $\vec{XY}$ and note that they point in opposite directions.
- **Ex:** Find the midpoint of the points (a,b) and (x,y).
  - Note that the midpoint is (a,b) plus half of $(x,y)-(a,b)$.

2 Geometry of $\mathbb{R}^n$

2.1 Lines in $\mathbb{R}^n$

- We're familiar with the equation of a line in $\mathbb{R}^2$.
- It looks like $y=mx+b$.
- Or a more general way of writing it is $ax+by=c$.
- At the beginning of the term, I could have asked something like: find the general solution to the system of equations $ax+by=c$
  - You would have done the following: construct the augmented matrix $(b \quad a \quad c)$.
  - Put it in RREF: $(1 \quad \frac{a}{b} \quad \frac{c}{b})$.
  - Assign a parameter to the free variable x, say t.
  - Then every solution looks like $x=t$, $y=\frac{c}{b}-\frac{a}{b}t$.
  - Rewriting this in vector form yields

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}t\\\frac{c}{b}-\frac{a}{b}t\end{pmatrix}=\begin{pmatrix}0\\\frac{c}{b}\end{pmatrix}+t\begin{pmatrix}1\\-\frac{a}{b}\end{pmatrix}$$

- We now know how to visualize this. Suppose $a = 2, b = 1, c = 3$ for this, so the equation of our line is
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ 3-2t \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$
- Then the line $2x + y = 3$ looks like the set of points you get by taking vector $(0, 3)$ and adding scalar multiples of $(1, -2)$.
- Of course, we already know this! $y = 3 - 2x$ has intercept 3 and slope -2!
- This is how you've been taught to draw lines since high school!
- But now we see that a "line" has the form $\vec{u} + t\vec{v}$ for some fixed vectors $\vec{u}$ and $\vec{v}$.

- This is true in 3 (and higher) dimensions, too!
- Equation of a Line: A line in $\mathbb{R}^n$ has the form $\vec{u}+t\vec{v}$ where $\vec{u}$ and $\vec{v}$ are fixed vectors in $\mathbb{R}^n$, $\vec{v} \neq \vec{0}$, and t ranges over all real numbers.

Def: In the equation of the line $\vec{u} + t\vec{v}$, the vector $\vec{v}$ is called the direction of the line.

- Observe: the vector $\vec{u}$ is always contained in the line $\vec{u} + t\vec{v}$. This corresponds to when $t = 0$.
- **Ex:** Find the line in $\mathbb{R}^3$ containing the points $P = (2, -1, 7)$ and $Q = (-3, 4, 5)$.
  -
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix} + t\begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}$$
  - The vector we're looking for is the direction vector
  - The direction vector points from one point to the other, hence, the direction vector is
$$\vec{PQ} = \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ -2 \end{pmatrix}$$
- **Ex:** Put the line
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$
into slope-intercept form.
  - Note that we have
$$x = 1 + 3t$$
$$y = 2 + 4t$$
  - Solve for t in terms of x: $t = \frac{x-1}{3}$
  - Replace the t in the second equation with $\frac{x-1}{3}$:
$$y = 2 + 4 \cdot \frac{x-1}{3} = \frac{2}{3} + \frac{4}{3}x$$
- **Ex:** Find the points of intersections between the lines
$$L_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \qquad L_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 1 \end{pmatrix} + s\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$
  - Set the vector equations equal to each other, solve a system of three equations and two variables.

- **Ex:** Find equations for the lines through $P = (1, 0, 1)$ which meet the line

$$L : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix}$$

at points distance 3 from $P_0 = (1, 2, 0)$.

  - Start by finding the points on L which are distance 3 from $(1, 2, 0)$
  - The unit direction vector is $\vec{u} = (2/3, -1/3, 2/3)$
  - Going 3 units from $(1, 2, 0)$ in the direction of $\vec{u}$ yields $(3, 1, 2)$.
  - Going 3 units from $(1, 2, 0)$ in the direction of $-\vec{u}$ yields $(-1, 3, -2)$.
  - Now the line between $(1, 0, 1)$ and $(3, 1, 2)$ is given by:

$$L_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}.$$

  - Now the line between $(1, 0, 1)$ and $(-1, 3, -2)$ is given by:

$$L_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}.$$

2.2 The Dot Product

2.2.1 Definition and Properties

- We've already seen the next object of study! Recall:

Def: Let $\vec{v} = (v_1, \ldots, v_n), \vec{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$. Then the dot/scalar/inner product of $\vec{v}$ and $\vec{u}$ is

$$\vec{v} \cdot \vec{u} = v_1 u_1 + v_2 u_2 + \cdots + v_n u_n.$$

Ex: Let $\vec{v} = (1, 2, 3)$. Compute $\vec{v} \cdot \vec{v}$.

- Note that $\vec{v} \cdot \vec{v}$ looks very familiar! In fact...
- Fact: for any $\vec{v} \in \mathbb{R}^n$, $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.

Q: How do dot products interact with addition and scalar multiplication?

- For all vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and for all numbers $k \in \mathbb{R}$:
  - $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
  - $(k\vec{v}) \cdot \vec{w} = k(\vec{v} \cdot \vec{w})$
  - $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.

2.2.2 Geometric Meaning

Q: What does the dot product of two vectors mean?

- Claim: the dot product tells us about the angle between two vectors via the following fact.

Def: For any two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, the included angle is the angle between the two vectors when drawn in standard position.

- Fact: For any two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ with included angle θ, $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta$.

- Proof:
  - Draw the triangle with sides $\vec{v}, \vec{w}$, and $\vec{v} - \vec{w}$.
  - Apply the law of cosines:

$$\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\| \cos\theta.$$

  - Rewrite $\|\vec{v} - \vec{w}\|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w})$
  - Expand, cancel terms, and voila!

Ex: What is the included angle between $\vec{v} = (1, 2, 3)$ and $\vec{w} = (2, -1, 1)$?

- Note: everything in radians.

Def: For vectors $\vec{v}, \vec{w}$ in $\mathbb{R}^n$, the <u>included angle</u> between $\vec{v}$ and $\vec{w}$ is the quantity

$$\arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right).$$

Ex: What is the included angle between $\vec{v} = (1, 1)$ and $\vec{w} = (-1, 1)$?

Def: Vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are <u>orthogonal/perpendicular</u> if $\vec{v} \cdot \vec{w} = 0$.

2.2.3 Projections

Ex: What is the distance from the line $3x + y = 0$ to the point $(1, 1)$?

  - Draw a perpendicular from the point to the line.
  - We want to know how long the perpendicular is.
  - We can do this by the following process: the perpendicular line has slope $1/3$
  - So we want the line that has slope $1/3$ and passes through $(1, 1)$.
  - This is the line $-x/3 + y = 2/3$
  - Where do these lines meet? $x = -1/5$ and $y = 3/5$
  - What's the distance between $(-1/5, 3/5)$ and $(1, 1)$? $\sqrt{40}/5$

- It would be nice to have an easier way of doing this problem.
- Consider what we're trying to do: make a right triangle.
- Here's the problem that we want to address: given vectors $\vec{v}$ and $\vec{w}$, can I make a right triangle whose hypotenuse is $\vec{v}$ and whose base is a *scalar multiple* of $\vec{w}$?
- Answer: yes, and we can do it with dot products!
- Here's the claim: the vector $\vec{p} = \frac{\vec{v} \cdot \vec{w}}{\|w\|^2}\vec{w}$ is a leg of a right triangle whose hypotenuse is $\vec{v}$.
- Proof:
  - The vector $\vec{p}$ and $\vec{v}$ certainly form a triangle whose third side is $\vec{v} - \vec{p}$.
  - What we need to check is that this is a right triangle with hypotenuse $\vec{v}$.
  - I.e. we need to check the angle between $\vec{p}$ and $\vec{v} - \vec{p}$.
  - So we use the dot product!

$$\vec{p} \cdot (\vec{v} - \vec{p}) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2}\vec{w} \cdot \left(\vec{v} - \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2}\vec{w}\right) = \frac{(\vec{v} \cdot \vec{w})^2}{\|w\|^2} - \frac{(\vec{v} \cdot \vec{w})^2}{\|w\|^4}\vec{w} \cdot \vec{w} = 0$$

- Hence, we have a right triangle and we are done.

- The vector $\vec{p}$ has a special name:

Def: Given any vector $\vec{v}$ and any nonzero vector $\vec{w}$, the projection of $\vec{v}$ onto $\vec{w}$ is defined to be

$$\operatorname{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|^2}\vec{w}.$$

- The projection points in the same direction as $\vec{w}$ and has the property that it forms the leg of a right triangle with $\vec{v}$.

Ex: What is the distance from the line $3x + y = 0$ to the point $(1, 1)$?

  - Note that we are looking for the projection of $\vec{v} = (1, 1)$ onto the vector $\vec{w} = (-1, 3)$
  - By the previous formula, we have

$$\operatorname{proj}_{\vec{w}}(\vec{v}) = \frac{2}{10}\begin{pmatrix}-1\\3\end{pmatrix} = \begin{pmatrix}-1/5\\3/5\end{pmatrix}.$$

  - We want the distance from $(1, 1)$ to $(-1/5, 3/5)$, which we have already found to be $\sqrt{40}/5$.

Ex: Find the distance from the point $(1, 1)$ to the line $3x + y = 1$.

  - Very similar except now we don't want to project the vector $(1, 1)$ onto the direction vector, we want to project something else.
  - We could project the vector from $(0, 1)$ to $(1, 1)$ onto the direction vector $(-1, 3)$.
  - To project $\vec{v} = (1, 0)$ onto $\vec{w} = (-1, 3)$, we get

$$\operatorname{proj}_{\vec{w}}(\vec{v}) = \frac{-1}{10}\begin{pmatrix}-1\\3\end{pmatrix} = \begin{pmatrix}1/10\\-3/10\end{pmatrix}.$$

  - So the point on the line closest to $(1, 1)$ is $(0, 1) + (1/10, -3/10) = (1/10, 7/10)$.
  - The distance between these points is then $\sqrt{81 + 27}/10 = \sqrt{108}/10$

- General process for finding the distance from a point P to a line L
  - Find a point on the line P_0
  - Project $\vec{P_0P}$ onto the direction vector of L.
  - The point on L closest to P is P_0+ the projection.
  - Find the distance between P and the point in the last step.

Ex: Find the minimum distance between the parallel lines $y = 2x + 1$ and $y = 2x - 5$.

  - We can start by picking any two points on our two lines: say $P_1 = (0, 1)$ and $P_2 = (0, -5)$.
  - Then, project the vector $\vec{P_1P_2} = (0, -6)$ onto the direction vector of either line, $\vec{d} = (1, 2)$:

$$\operatorname{proj}_{\vec{d}}(\vec{P_1P_2}) = \frac{-12}{5}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-12/5\\-24/5\end{pmatrix}.$$

  - The difference between $\vec{P_1P_2}$ and the projection we just found gives the vector of interest: $(12/5, -6/5)$.
  - The length of this vector (and hence, the distance between the lines) is $\frac{\sqrt{180}}{5}$.

2.3 Planes

- Recall that in $\mathbb{R}^2$, there were two forms of a line: an equation $ax + by = c$, and a parametric form $\begin{pmatrix} x \\ y \end{pmatrix} = \vec{a} + t\vec{d}$.
- The latter form is what generalized the most easily.
- Next up: planes
- What is a plane?
- Maybe we should first mention (informally) what a linear space is: a linear space is the set of solutions to some system of linear equations. The dimension of a linear space is the number of parameters needed to describe the solutions.
- There are two very reasonable definitions of a plane:
  - A plane could be a two dimensional space living inside of a higher dimensional space
  - A plane could also be an $n-1$-dimensional space living inside of an n-dimensional space.
- These definitions coincide in $\mathbb{R}^3$ (which is the most common setting), but diverge in other dimensions.
- We're going to take the latter definition.
- What does an $n-1$-dimensional space look like?
- It should have $n-1$ parameters, meaning it should be described by a single equation!
- What is that equation?
- $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$.
- Okay, that's a really boring answer, but it works.

Ex: Find an equation of the plane which passes through the points $(1, 2, 3)$, $(1, 0, 1)$, and $(2, 1, 1)$.

- We are trying to solve the system

$$\begin{aligned} a_1 + 2a_2 + 3a_3 &= b \\ a_1 + a_3 &= b \\ 2a_1 + a_2 + a_3 &= b \end{aligned}$$

- You can put the augmented matrix in RREF, or you can write the system as $AX = B$, invert A, and have $X = A^{-1}B$.
- Either way, you find that $a_1 = b/2$, $a_2 = -b/2$, and $a_3 = b/2$.
- So there are many equations of this plane! We just need to pick a (nonzero) value of b, and we'll get an equation of our plane.

Q: How else can we find an equation for a plane?

- There should be a vector, $\vec{n} = (a, b, c)$, which is orthogonal to every vector "in" the plane.

Def: A vector $\vec{n}$ which is orthogonal to every vector in plane is called a <u>normal</u> vector to that plane.

- What do we mean by this?
- We want to describe the set of all points $Q = (x, y, z)$ in our plane.
- Let's say that the plane has some point, $P_0 = (x_0, y_0, z_0)$, which we know.

- Then the vectors in the plane all have the form $\vec{P_0Q}$.
- Then $\vec{n}$ is orthogonal to $\vec{P_0Q}$ no matter which Q we pick.
- This means that $\vec{n} \cdot \vec{P_0Q} = 0$.
- Now let's write these things in component form:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \right] = 0.$$

- Rearrange to get $ax + by + cz = ax_0 + by_0 + cz_0$.
- Let $d = ax_0 + by_0 + cz_0$, so that we get $ax + by + cz = d$.
- What does this mean? The coefficients of the equation tell us a normal vector to the plane!
- And the value of d is determined by a point in that plane!

Ex: Find the equation of the plane with normal vector $(4, 0, -1)$ and which passes through the point $(1, 2, 3)$.

- Solution: $4x - z = 4 \cdot 1 - 3 = 1$. So $4x - z = 1$.

Ex: Find distance from the point $P = (1, 2, 3)$ to the plane $2x + y - z = 3$.

  - We want to find the vector with one endpoint at $(1, 2, 3)$, with the other endpoint in the plane, and which is perpendicular to the plane. (Draw picture.)
  - I.e. we want to take any vector from the plane to the point, then project it onto the normal vector of the plane.
  - The point $P_0 = (0, 0, -3)$ is on the plane.
  - So the vector $\vec{v} = \vec{P_0P} = (1, 2, 6)$ passes from the plane to the point.
  - To project $\vec{v} = (1, 2, 6)$ onto $\vec{n} = (2, 1, -1)$, we get

$$\text{proj}_{\vec{n}}(\vec{v}) = \frac{\vec{v} \cdot \vec{n}}{\|\vec{n}\|^2} \vec{n} = \frac{-2}{6} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4/3 \\ -1/3 \\ 1/3 \end{pmatrix}$$

  - The length of this vector is $\frac{1}{3} \cdot \sqrt{6}$, so that's the distance from the line to the plane.

3 The Cross Product

Ex: Find an equation of the plane which passes through the points $P_1 = (1, 2, 3)$, $P_2 = (1, 0, 1)$, and $P_3 = (2, 1, 1)$.

- This example was kind of annoying.
- Now that we know something about normal vectors to planes, it would be great if we could find a normal vector to the plane containing these points.
- A normal vector to the plane would be perpendicular to both $\vec{P_1P_2} = (0, -2, -2)$ and $\vec{P_1P_3} = (1, -1, -2)$.

Q: Given two vectors, $\vec{v}$ and $\vec{w}$, how do we find a vector that is perpendicular to each of them?

- In some sense, this question is easy: find any solution to $(x, y, z) \cdot (0, -2, -2) = 0$ and $(x, y, z) \cdot (1, -1, -2) = 0$

- There's going to be a cute way to do this in three dimensions, however.

Def: Let $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$, and $\vec{k} = (0, 0, 1)$.

Def: The cross product of vectors $\vec{v} = (a, b, c)$ and $\vec{w} = (x, y, z)$ is

$$\vec{v} \times \vec{w} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ x & y & z \end{pmatrix}.$$

- Note: this is not a proper determinant, since the top row consists entirely of vectors, not numbers! But we can still do the same operations as if everything made sense.
- Note: $\vec{v} \times \vec{w}$ is a vector, though this is not obvious yet.
- Note: this operation only makes sense if you start with two vectors in $\mathbb{R}^3$. Try extending this definition to other dimensions to see why it doesn't work.

Ex: Let $\vec{v} = (1, 2, 3)$ and $\vec{w} = (0, 1, 1)$. Find $\vec{v} \times \vec{w}$. What is the angle between $\vec{v}$ and $\vec{v} \times \vec{w}$? What is the angle between $\vec{w}$ and $\vec{v} \times \vec{w}$?

  -
$$\vec{v} \times \vec{w} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix} = \vec{i}(2-3) - \vec{j}(1-0) + \vec{k}(1-0) = -\vec{i} - \vec{j} + \vec{k} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$
  - Now we compute angles and note that $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$!
  - Magic.

- Properties of $\vec{v} \times \vec{w}$:
  - $\vec{v} \times \vec{w}$ is always orthogonal to both $\vec{v}$ and $\vec{w}$.
  - $\|\vec{v} \times \vec{w}\| = \|\vec{v}\|\|\vec{w}\| \sin\theta$ where θ is the included angle of $\vec{v}$ and $\vec{w}$.
  - $\vec{v}, \vec{w}$, and $\vec{v} \times \vec{w}$ form a right-handed system.
  - $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.
  - $\vec{v} \times \vec{v} = \vec{0}$
  - $(k\vec{v}) \times \vec{w} = k(\vec{v} \times \vec{w}) = \vec{v} \times k\vec{w}$.
  - $\vec{v} \times (\vec{u} + \vec{w}) = \vec{v} \times \vec{u} + \vec{v} \times \vec{w}$.
  - $(\vec{v} + \vec{u}) \times \vec{w} = \vec{v} \times \vec{w} + \vec{u} \times \vec{w}$.
- From these properties, we can see that the cross product is also kind of like multiplying vectors in $\mathbb{R}^3$.
- What else can the cross product tell us?
- Recall this magic fact: $\|\vec{v} \times \vec{w}\| = \|\vec{v}\|\|\vec{w}\| \sin\theta$
- Note that the RHS is also the area of the parallelogram whose sides are determined by $\vec{v}$ and $\vec{w}$.

Ex: Find the area of the triangle whose vertices are at $P_1 = (1, 2, 3)$, $P_2 = (-1, 0, 1)$, and $P_3 = (2, -1, 0)$.

  - The sides of this triangle are the vectors $\vec{v} = \vec{P_1P_2} = (-2, -2, -2)$ and $\vec{w} = \vec{P_1P_3} = (1, -3, -3)$.
  - Now take half of the magnitude of the cross product.

Ex: Find the equation of the plane which includes the points $P = (1, 2, 3)$ and the line

$$L : \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + t \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}.$$

- Take the vector from P to some point on the line and cross it with the direction vector of the line to get the normal to the plane.
- Then put together the normal vector with the point and call it a day.

TPS: Which of the following statements are true?

1. If P is a plane and Q is a point in the plane, then the vector from the origin to Q is orthogonal to the normal vector of P.
2. If P is a plane and Q_1 and Q_2 are points in the plane, then the vector from Q_1 to Q_2 is orthogonal to the normal vector of P.
3. If P is a point and L is a line, then there is a unique plane containing P and L.

4 The Box/Triple Product

Def: Let $v, w, x \in \mathbb{R}^3$ be nonzero vectors which do not all lie in the same plane. The parallelepiped spanned by v, w, x is the polyhedron with vertices at $0, v, w, x, v+w, v+x, w+x$, and $v+w+x$.

- It's called a parallelepiped because each side is a parallelogram.
- Fact: the volume of the parallelepiped spanned by v, w, x is equal to

$$|v \cdot (w \times x)|.$$

- Cool fact: order doesn't matter here. You can cross your favorite two vectors, and dot with the third. Doesn't matter which ones you pick.