Chapter 1: Systems of Equations

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January 9, 2024

1 Equations And Solutions

- In middle school, you learn how to solve one-variable linear equations: $5x - 1 = 2$
- There are two ways to make this more complicated:
  - Make it nonlinear: $5x\cos(x) - 1 = 2$ (calculus)
  - Add more variables and equations: $5x - y = 2$, $10x + y = 1$ (linear algebra)
- We will focus on the second one.

Def: A linear equation has the form
$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$
for $a_1, \ldots, a_n, b \in \mathbb{R}$ and variables $x_1, \ldots, x_n$.

Def: The a_i are called coefficients, the x_i are called variables, and b is called the constant.

Def: A system of linear equations is a collection of at least one linear equation.

Ex:
$$5x - y = 2$$
is a system of linear equations.

Ex:
$$\begin{aligned} 5x - y &= 2 \\ 10x + y &= 1 \end{aligned}$$
is a system of linear equations.

- What do we like to do when we have equations? Solve them.
- But what does "solve" mean? Find every solution.

Ex: Solve
$$\begin{aligned} 5x - y &= 2 \\ 10x + y &= 1 \end{aligned}$$

- Method 1: $y = 5x - 2$, then sub and solve
- Method 1 is not amazing because we got lucky in that solving for y was easy.
- Method 2: $-2r_1 + r_2$, then solve for y.
- Method 2 works out to be more reliable, but we'll revisit this.

2 Solution Geometry

- Let's take a quick look at how we can visualize solutions to linear systems of equations.
- One variable: $ax = b$ corresponds to a point on the real line: $x = b/a$ OR it has no solutions (if $a = 0$ and $b \neq 0$) OR it has infinitely many solutions (if $a = 0$ and $b = 0$).
- Two variables:

$$\begin{aligned} 5x - y &= 2 \\ 10x + y &= 1 \end{aligned}$$

TPS: What geometric object is represented by the equation $5x - y = 2$?

- A linear equation like $5x - y = 2$ has infinitely many solutions. We represent those solutions as a line in the plane.
- Draw each line.
- A system of equations in two variables can be represented by a plane with a line for each equation.
- A solution to the system of equations then is an intersection point of all of those lines.
- Make sure to draw some systems with three or more equations.
- This leads to three possibilities: no solutions, one solution, or infinitely many solutions.

- Three variables: This is for later in the course, but each equation will represent a plane. There will again be three possibilities for the solution(s): none, one, or infinitely many.

3 Solution Classification and Gaussian Elimination

Thm: A system of linear equations has zero, one, or infinitely many solutions.

Def: A system of linear equations is <u>consistent</u> if it has at least one solution. It is <u>inconsistent</u> if it has no solutions.

- Let's see an example of each and generate a process for solving these equations

Ex:

$$\begin{aligned} 2x + 4y - z &= 2 \\ -x + 3y + z &= 0 \\ y - z &= 1 \end{aligned}$$

has a unique solution.

- Solve by first getting the x-coeff to be 1, then clear out the xs below it
- Then get the y-coeff to be 1, then clear out the ys below it
- Then get the z-coeff to be 1 and back substitute

Def: The process of "get the x-coeff to be 1, then clear. Now look at the smaller matrix below it." is called <u>Gaussian elimination</u>.

Ex:

$$\begin{aligned} 4y - z &= 2 \\ -x + 3y + z &= 0 \\ -3x + 17y + z &= 5 \end{aligned}$$

has no solutions (since one equation reduces to $0 = 1$, more or less)

- Note that Gaussian elimination has us rearrange two equations. This is not the only way to do it, but it is a way and it's the way we're learning.

Ex:

$$\begin{aligned} 4y - z &= 2 \\ -x + 3y + z &= 0 \\ -3x + 17y + z &= 4 \end{aligned}$$

has infinitely many solutions.

- Give a parameter to z

TPS: We've done lots of writing so far. What's something we could do in order to spend less time writing? Be creative!

4 Saving Ink: Matrices

- We've been doing a lot of writing so far and much of it is redundant.
- Goal: save ink and write things more efficiently.
- We can do everything that we want to do by only keeping track of the coefficients and constants.
- We've also only been doing three operations so far:

Def: The three <u>elementary operations</u> are:

1. Swap two equations.
2. Multiply an equation by a nonzero constant (or <u>scalar</u>).
3. Add a constant multiple of one equation to another.

Thm: Elementary operations preserve solutions.

- This is a fact we already know, but it's worth mentioning.
- Do each of the three prior examples side-by-side with augmented matrices.
  - Make sure to define the <u>augmented</u>, <u>coefficient</u>, and <u>constant</u> matrices of the systems.

Def: The three <u>elementary row operations</u> are:

1. Swap two rows.
2. Multiply a row by a nonzero scalar.
3. Add a scalar multiple of one row to another.

Def: A matrix is in <u>row-echelon form</u> if *all* of the following hold:

- Every row of zeroes is at the bottom of the matrix.
- Every leading entry in a nonzero row is 1 (this 1 is called a <u>pivot</u> or a <u>leading one</u>).
- Every pivot is to the right of the pivots above it.

Ex: In REF:

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Ex: Not in REF:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

TPS: True or false? A matrix in REF can have a nonzero entry directly below a pivot.

Def: A matrix is in reduced row-echelon form if *all* of the following hold:

- The matrix is in REF.
- Every entry above every pivot is 0.

Ex: In RREF:

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Ex: Not in RREF:

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

TPS: Come up with a matrix with two rows and two columns, which is in RREF, and which is not $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

- Why do we care about RREF?
- It makes the solutions to systems of equations easier

Ex: Find all solutions to the system of equations by putting its augmented matrix in RREF.

$$\begin{aligned} -x_1 + 2x_2 - x_4 &= 1 \\ x_2 - x_3 + 3x_4 &= 0 \\ -x_1 + 2x_3 - 7x_4 &= 1 \end{aligned}$$

Def: The process of putting a matrix in RREF is called Gauss-Jordan elimination or row reduction.

Def: Suppose you have a linear system of equations in the variables $x_1, \ldots, x_n$. The variable x_i is said to be a pivot or basic variable if there is a pivot in its column in the REF of the augmented matrix of the system. Otherwise, x_i is a free variable.

Ex: Find the basic and free variables of the system

$$\begin{aligned} -x_1 + 2x_2 - x_4 &= 1 \\ x_2 - x_3 + 3x_4 &= 0 \\ -x_1 + 2x_3 - 7x_4 &= 1 \end{aligned}$$

Thm: The RREF of a matrix is unique!

- Note: there are many correct ways to get there, but only one RREF.

Q: Can we tell, from the RREF of the augmented matrix, if a system has zero, one, or infinitely many solutions?

- Yes!
- Zero solutions: the last column has a pivot

- One solution: every column except the last column has a pivot
- Infinitely many solutions: the last column does not have a pivot and some variable is free.

- We can express this in terms of a quantity called the rank of a matrix.

Def: The rank of a matrix is the number of pivot columns in its RREF.

Ex: The rank of

$$\left(\begin{array}{cccc|c} -1 & 2 & 0 & -1 & 1 \\ 0 & 1 & -1 & 3 & 0 \\ -1 & 0 & 2 & -7 & 1 \end{array}\right)$$

is 2 because its RREF has 2 pivot columns.

Ex: The rank of

$$\begin{pmatrix} a & b & 5 \\ 1 & -2 & 1 \end{pmatrix}$$

is 1 if $a = 5, b = -10$ and 2 otherwise.

- Solution classification in terms of rank:
  - Suppose the augmented matrix for a system of linear equations has n columns.
  - Zero solutions: the last column has a pivot
  - One solution: no pivot in the last column and the augmented matrix has rank $n - 1$
  - Infinitely many solutions: no pivot in the last column and the augmented matrix has rank $< n-1$.

5 Homogeneous Systems

- A common type of linear system of equations has all of the constants equal to 0.

Def: A linear system of equations where all of the constants are 0 is called <u>homogeneous</u>.

Ex:

$$\begin{aligned} 3x - z &= 0 \\ 2x + y &= 0 \end{aligned}$$

is a homogeneous system of linear equations.

- Why do we care about these?
  - They show up in a lot of places (differential equations and population modeling, in particular).
  - They are a little simpler than non-homogeneous solutions.

- Observe that a homogeneous system of linear equations always has a solution: $x_1 = 0, \ldots$

- Hence, homogeneous systems are always consistent.

Def: The solution $x_1 = 0, \ldots$ is called the <u>trivial</u> solution to a system of linear equations.

- Note that this means homogeneous systems only have two possibilities for their solutions: one solution or infinitely many solutions.