Chapter 5: Linear Transformations

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1 Introduction

- One of the purposes of mathematics is to model things we see in the real world.
- What happens in the real world? Things change!
- If you have a chemical reaction occurring in a big vat, then the temperature of the reaction is a function of a bunch of things:
  - Which point you pick in the vat
  - How long the reaction has been proceeding
  - Which chemicals you're combining.
- So we want to be able to analyze functions with many inputs
- Sometimes, we also want to analyze functions with many outputs.
- For example, think about my position in the world as a function of time.
- That function has three dimensions worth of outputs because we live in a three dimensional space.
- Notation: we denote a function which has n inputs and m outputs with the notation $T : \mathbb{R}^n \to \mathbb{R}^m$. T is the name of the function, it takes inputs which live in $\mathbb{R}^n$, and outputs vectors which live in $\mathbb{R}^m$.

Ex: Consider the function $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^x \\ x^2y^3 \\ \ln(y^2+1) \end{pmatrix}.$$

- Instead of writing $T(\vec{v})$ like we usually do in calculus, sometimes, we just write $T\vec{v}$.
- Now, the above example of a function is very complicated.
- We don't have enough background to just study T right away, so we're going to start with the "simplest" functions there are: linear functions.

Def: A linear function/linear transformation/linear map is any function $T : \mathbb{R}^n \to \mathbb{R}^m$ which satisfies the following two properties:

- $T(\vec{v} + \vec{w}) = T\vec{v} + T\vec{w}$ for every $\vec{v}, \vec{w} \in \mathbb{R}^n$.
- $T(c\vec{v}) = cT\vec{v}$ for every $c \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^n$.

Ex: Is the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 3x + 4y \end{pmatrix}$$

a linear transformation?

Ex: Is the function $T : \mathbb{R} \to \mathbb{R}$ defined by $Tx = 2x + 1$ a linear transformation? What about $U : \mathbb{R} \to \mathbb{R}$ defined by $Ux = 2x$?

Ex: Is the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

a linear transformation?

- General fact: if A is an $m \times n$ matrix, then $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T\vec{v} = A\vec{v}$ is a linear transformation.

TPS: Which of the following are true?

- Every function $\mathbb{R}^2 \to \mathbb{R}^2$ is linear.
- If $T : \mathbb{R} \to \mathbb{R}$ is defined by $T(x) = kx$ for some constant k, then T is linear.
- If $T : \mathbb{R} \to \mathbb{R}$ is linear, then there exists a constant k so that $T(x) = kx$.

- Aside: let's learn to visualize matrix transformations.
- Functions that you are used to graphing have one input and one output.
- It's possible for us to graph functions with two inputs and one output.
- But we can't just draw a graph of functions with two inputs and two outputs.
- So instead, we need to be clever: `https://www.geogebra.org/m/YCZa8TAH`
- Try some matrices and just kind of see what happens.

Ex: Is the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by "$T\vec{v} =$ rotate $\vec{v}$ by $\pi/2$" a linear map?

- Draw the picture.
- Alternatively, we could give a matrix!
- Go to `https://www.geogebra.org/m/YCZa8TAH` and use the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Ex: Let

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \quad \text{and} \quad \vec{w} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix}.$$

If $T : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation with

$$T\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad T\vec{w} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

what is $T(2\vec{v} - \vec{w})$?

2 Matrices for Linear Transformations

Ex: Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with

$$T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \text{and} \quad T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

1. What is $T(2, 0)$?

2. What is $T(1,1)$?
3. What is $T(-100,3)$?
4. What is $T(x,y)$?

- Note that this exercise could be repeated easily if I change up the numbers: if you know $T(1,0)$ and $T(0,1)$, you can write a formula for $T(x,y)$!
- More than that: if you have a linear function T and you know $T(1,0)$ and $T(0,1)$, then T has a matrix and its first column is $T(1,0)$ and its second column is $T(0,1)$.
- This turns out to generalize very nicely.
- Notation: For any n and any $1 \leqslant i \leqslant n$, let $\vec{e_i}$ denote the ith column of I_n.
- This is a silly way of saying that e_i denotes the vector with a 1 in the ith entry and 0 everywhere else.
- Fact: if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then for every $\vec{v} \in \mathbb{R}^n$, $T\vec{v} = A\vec{v}$ where

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}.$$

Ex: Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ which has

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} \quad \text{and} \quad T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\0\\-2\end{pmatrix}.$$

Ex: Let $\vec{w} = \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right) \in \mathbb{R}^2$. Is function $T : \mathbb{R}^2 \to \mathbb{R}^2$ which has $T\vec{v} = \mathrm{proj}_{\vec{w}}(\vec{v})$ a linear map? If so, what is its matrix?

Ex: Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ which has

$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} \quad \text{and} \quad T\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}-1\\0\\-2\end{pmatrix}.$$

- Goal: figure out $T(1,0)$ and $T(0,1)$.
- To do this, write $(1,0)$ as a linear combination of $(1,1)$ and $(-1,2)$.
- $(1,0) = \frac{2}{3}(1,1) - \frac{1}{3}(-1,2)$
- Also, write $(0,1)$ as a linear combination of $(1,1)$ and $(-1,2)$.
- $(0,1) = \frac{1}{3}(1,1) + \frac{1}{3}(-1,2)$.
- Then compute $T(1,0)$ and $T(0,1)$.

- Notice that writing $(1,0)$ as a linear combination of $(1,1)$ and $(-1,2)$ is equivalent to solving

$$\begin{pmatrix}1 & -1\\1 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}.$$

- Likewise, writing $(0,1)$ as a linear combination of $(1,1)$ and $(-1,2)$ is equivalent to solving

$$\begin{pmatrix}1 & -1\\1 & 2\end{pmatrix}\begin{pmatrix}w\\z\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}.$$

- We can solve these two systems at the same by solving

$$\begin{pmatrix}1 & -1\\1 & 2\end{pmatrix}\begin{pmatrix}x & w\\y & z\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}.$$

- Oh wait, that's matrix inversion!
- What did we do to the inverse matrix to get the matrix of T? We applied x to $T(1,1)$ and y to $T(-1,2)$ to get the first column of the matrix. Likewise for the other column.
- Leap: we multiplied the matrix whose columns were $T(v_1)$ and (v_2) by the inverse of the matrix whose columns are v_1 and v_2.
- General process: Suppose that $v_1, \ldots, v_n \in \mathbb{R}^n$ and suppose that the matrix

$$V = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

is invertible. If T is a linear map with $Tv_1 = b_1, \ldots, Tv_n = b_n$, then the matrix of T is given by

$$\begin{pmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}^{-1}.$$

TPS: Which of the following are true?

1. The map $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} x+y \\ xy \end{smallmatrix}\right)$ is linear.
2. If $T : \mathbb{R}^3 \to \mathbb{R}$ is a linear map, then $T(\vec{v} \times \vec{w}) = T(\vec{v})T(\vec{w})$ for every $\vec{v}$ and $\vec{w}$.
3. If $v_1, \ldots, v_n \in \mathbb{R}^n$, $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, and you know $T(v_1)$, $T(v_2)$,...,$T(v_n)$, then you can compute $T(v)$ for any $v \in \mathbb{R}^n$.

- Sometimes, we don't have enough information to construct the full matrix, but this can turn out to be okay.

Ex: Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is linear and $T(1,0,0) = (1,1)$, and $T(1,1,0) = (3,4)$. What is $T(3,7,0)$? What is $T(1,0,1)$?

3 Special Linear Transformations

- There are two especially interesting types of linear transformations: rotations and reflections.

Ex: Find the matrix of the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ which rotates any vector by an angle θ.

- $Te_1 = (\cos\theta, \sin\theta)$
- $Te_2 = (-\sin\theta, \cos\theta)$

Ex: Find the matrix of the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by reflection across the line $y = mx$.

- We'll just do Te_1 in class.
- Draw the right triangle with vertices $(1,0)$, $(0,0)$, $(1,m)$.
- Label the angle at the origin as θ
- Then we know that $\cos\theta = \frac{1}{\sqrt{1+m^2}}$ and $\sin\theta = \frac{m}{\sqrt{1+m^2}}$.
- We also know that Te_1 is a rotation of e_1 by an angle 2θ.
- So $Te_1 = (\cos(2\theta), \sin(2\theta)) = (\cos^2\theta - \sin^2\theta, 2\sin\theta\cos\theta)$.
- Now write everything in terms of m and call it a day.
- End result: the matrix of T is

$$\frac{1}{1+m^2} \begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$$

4 Composition and Inversion

- We can do a bunch of things with matrices, like add, scale, multiply, and sometimes divide them.
- What do these matrix operations mean in terms of the corresponding functions.
- Addition and scaling are easy: you can add and scale functions!
- Matrix multiplication is tricky however.
- Let's say you have a function $T : \mathbb{R}^n \to \mathbb{R}^m$ and a function $S : \mathbb{R}^m \to \mathbb{R}^p$.
- Let's say that T has some matrix A (note that A is $m \times n$) and S has a matrix B (note that B is $p \times m$).
- What function does BA correspond to?
- It corresponds to $x \mapsto BAx$, i.e. first multiply by A, then by B, i.e. first apply T, then apply S.
- This is what we mean by function composition.
  - Note that $T \circ S$ means "first S then T."

Ex: What is the matrix of the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ formed by first reflecting across the line $y = 2x$ and then rotating counterclockwise by $\pi/3$?

Ex: What is the matrix of the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ formed by first rotating counterclockwise by θ then rotating counterclockwise by ψ?

- First: note that this is one rotation by $\theta + \psi$
- Second: note that this is function composition, so multiply the appropriate matrices.
- We recover trig identities!

- What about matrix inversion?
- Let A be an $n \times n$ invertible matrix which corresponds to the linear map $T : \mathbb{R}^n \to \mathbb{R}^n$.
- Then A^{-1} corresponds to a linear map $S : \mathbb{R}^n \to \mathbb{R}^n$.
- How do T and S interact?
- If I do $T \circ S$, that corresponds to $AA^{-1} = I_n$, which corresponds to the identity map.
- Likewise with $S \circ T$.
- We find that matrix inverses correspond to function inverses.
- The inverse matrix "undoes" the original matrix.

Ex: What is the inverse of the rotation by θ matrix?

- We find this by noting that we can "undo" the action by rotating by $-\theta$ and then writing down that matrix.
- We don't have to invert the rotation matrix.

TPS: Which of the following are true?

1. The composition of two linear maps must be a linear map.
2. If a linear map has an inverse, then the inverse must be a linear map.
3. Every linear map has an inverse.

- Note: the phrase "T maps U to B" means "$T(U) = B$."

Ex: Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map whose matrix is

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Find a vector U so that T maps U to $(1, 0, 1)$.

Ex: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation so that

$$T\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \text{ and } T\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Compute $T^{-1}\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$.

- Write $(0, 1)$ as a linear combination of $(3, 3)$ and $(-1, 0)$.
- Then apply T^{-1} to both sides.

5 Exam Review

Ex: Which of the following are true?

- If A is an $n \times n$ matrix, then $\det(\operatorname{adj} A) = \det(A)^{n-1}$.
  - True because $A(\operatorname{adj} A) = \det(A)I_n$.
- Let $\vec{v}, \vec{w} \in \mathbb{R}^n$. Suppose that $\|\vec{w}\| = 1$ and $\operatorname{proj}_{\vec{w}}(\vec{v})$ has length $\frac{1}{2}$. Then $\vec{v} \cdot \vec{w} = \frac{1}{2}$.
  - False because we could have $\vec{w} \cdot \vec{v} = \pm\frac{1}{2}$.
- Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map and that

  $$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ and } T\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

  Then the matrix of T must be $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.
  - True because you know T on a basis of $\mathbb{R}^2$.

Def: Let $n \geqslant 3$. Two lines L_1 and L_2 are called <u>skew</u> lines if they do not intersect, but are not parallel.

- Notation: the phrase "the point $A(1, 1, 1)$" means the same thing as "the point $A = (1, 1, 1)$."
- Rotations are understood to always be counterclockwise (unless otherwise specified)
- Know your unit circle!