Chapter 2: Matrices

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1 Matrix Addition and Scalar Multiplication

1.1 The Basics

- Big long-term goal: look at the matrix of a linear system and quickly decide how many solutions it has.
- RREF takes a long time!!!
- To achieve this, we need to view the matrix as a whole, rather than a list of parts.
- The way we do this is by having matrix operations.
- We treat matrices similar to how we treat numbers.
- The first thing we'll start with is matrix addition.
- We want this to be very similar to numerical addition and this is easy to accomplish.
- The way to add two matrices is to add their corresponding components.

Ex:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -3 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

- If two matrices are different sizes, they cannot be added!
- Now recall how multiplication of numbers works:

$$\begin{aligned} 2 \cdot 5 &= 5 + 5 = 10 \\ 3 \cdot 5 &= 5 + 5 + 5 = 15 \end{aligned}$$

  etc.

- Scalar multiplication of matrices works similarly:

$$\begin{aligned} 2 \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 & 2 \cdot 2 \\ 2 \cdot 3 & 2 \cdot 4 \end{pmatrix} \\ 3 \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix} = \begin{pmatrix} 3 \cdot 1 & 3 \cdot 2 \\ 3 \cdot 3 & 3 \cdot 4 \end{pmatrix} \end{aligned}$$

  etc.

- In general: to multiply a matrix by the number c, multiply each of its entries by c.

Ex:

$$-1.5 \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} -3 & 1.5 \\ -1.5 & -6 \end{pmatrix}$$

- Multiplying two matrices together is trickier and we'll come back to it later.

1.2 The Notation

- Describing addition and scalar multiplication isn't so bad, but we'll find it helpful later to have a little notation.

Def: The <u>dimension</u> or <u>size</u> of a matrix is $m \times n$ if it has m rows and n columns.

- General mantra: rows by columns

Ex: Do a 3×2 (A) and a 2×3 matrix (B).

Def: The <u>(i, j) entry</u> in a matrix is the number in row i and column j (counting from the upper left entry).

Ex: Use one of the previous examples

- Common notation: the notation

$$A = (a_{ij})_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}}$$

means

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- Let's rewrite our addition rule using our new notation: Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \; B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

and let $c \in \mathbb{R}$. Then $A + B = (a_{ij} + b_{ij})$ and $cA = (ca_{ij})$.

1.3 Some Algebraic Rules

- The algebraic rules for matrix addition and scalar multiplication work exactly like you expect them to. Let A, B, C be $m \times n$ matrices and let $k, p \in \mathbb{R}$.
  - Addition is associative: $(A + B) + C = A + (B + C)$
  - Addition is commutative: $A + B = B + A$
  - An additive identity exists: $A + 0 = A$
  - Additive inverses exist: there exists a matrix D so that $A + D = 0$.
  - Scalar multiplication distributes: $(k + p)A = kA + pA$ and $k(A + B) = kA + kB$

Ex: Find the matrix A if

$$2(A + \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}) + \begin{pmatrix} -2 & -1 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

1.4 Revisiting Systems of Equations

- Recall the system of equations

$$\begin{aligned} -x_1 + 2x_2 - x_4 &= 1 \\ x_2 - x_3 + 3x_4 &= 0 \\ -x_1 + 2x_3 - 7x_4 &= 1 \end{aligned}$$

  from last section.

- The solutions to this had the form

$$\begin{aligned} x_1 &= -1 + 2s - 7t \\ x_2 &= s - 3t \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

  for $s, t \in \mathbb{R}$.

- We can conveniently rewrite this in matrix notation and use our new matrix arithmetic rules:

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 + 2s - 7t \\ s - 3t \\ s \\ t \end{pmatrix} &= \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2s \\ s \\ s \\ 0s \end{pmatrix} + \begin{pmatrix} -7t \\ -3t \\ 0t \\ t \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

Def: A matrix with one row or one column is called a vector.

Def: Identify the basic solutions and the particular solution in the example above.

- Note: the particular solution is a solution to the system we started with.
- Note: the basic solutions are not solutions to the system we started with.
- Why are they called basic solutions? Because they are solutions to the homogenization of the system we started with.

Def: Given vectors $v_1, \dots, v_n$ (each with m rows), a linear combination of $v_1, \dots, v_n$ is any vector of the form

$$c_1 v_1 + \cdots + c_n v_n.$$

- Rephrasing something we already know: the solutions to a linear system of equations can be written as a particular solution plus a linear combination of basic solutions.

Ex: A linear combination of $(1,1)$ and $(2,-1)$ is $1(1,1) + (-3)(2,-1) = (-5,4)$. Another linear combination of those vectors is $-(1,1) + 2(2,-1) = (3,-3)$.

Ex: Can $(1,2,3)$ be written as a linear combination of the vectors $(1,0,1), (1,-1,0)$, and $(5,-3,2)$?

  - The RREF of the augmented matrix is

$$\begin{pmatrix} [ccc|c]1 & 0 & 2 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so for example,

$$\begin{pmatrix}1\\2\\3\end{pmatrix} = 3\begin{pmatrix}1\\0\\1\end{pmatrix} - 2\begin{pmatrix}1\\-1\\0\end{pmatrix} + 0\begin{pmatrix}5\\-3\\2\end{pmatrix} = 1\begin{pmatrix}1\\0\\1\end{pmatrix} - 5\begin{pmatrix}1\\-1\\0\end{pmatrix} + 1\begin{pmatrix}5\\-3\\2\end{pmatrix}.$$

Ex: Can $(0, 0, 7)$ be written as a linear combination of $(1, 0, 0)$ and $(0, 1, 0)$?

Ex: Express the system of linear equations

$$\begin{aligned} 2x_1 - 4x_2 + x_4 &= 1 \\ 3x_2 - x_3 + x_4 &= -2 \\ -x_1 - x_2 - x_4 &= -1 \end{aligned}$$

as a vector equation.

2 Transposition

- Another useful thing that we can do is reflect matrices.
- It's not obvious why this is helpful, and we won't really get into it in this course!
- But we're doing it anyways.

Def: If $A = (a_{ij})$ is an $m \times n$ matrix, then the <u>transpose</u> of A is the $n \times m$ matrix $A^\mathsf{T} = (a_{ji})$.

- In other words, we reverse the roles of the rows and columns.

Ex: Do A, A^T, and $(A^\mathsf{T})^\mathsf{T}$ to a nonsquare example.

- Two useful properties. Let A and B be $m \times N$ matrices and let $r, s \in \mathbb{R}$.
  - $(A^\mathsf{T})^\mathsf{T} = A$
  - $(rA + sB)^\mathsf{T} = rA^\mathsf{T} + sB^\mathsf{T}$

Ex: Find A if

$$(2A + \begin{pmatrix}1 & -2\\2 & 3\end{pmatrix})^\mathsf{T} = \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}. \tag{1}$$

Def: The matrix A is <u>symmetric</u> if $A = A^\mathsf{T}$. It is <u>skew-symmetric</u> if $A = -A^\mathsf{T}$.

Ex: One symmetric and one skew-symmetric.

Ex: square matrix that is neither, symmetric matrix from previous example is not skew-symmetric, skew-symmetric matrix from the previous example is not symmetric.

3 Matrix Multiplication

3.1 How To and Why

- Things we can do: add matrices together, multiply numbers by matrices.
- A thing we would like to do: multiply matrices together.
- The plan: I tell you how, then I (partially) tell you why.

Def: Given two vectors $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$ *of the same size*, the dot product of v and w is

$$v \cdot w = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

- Note: the dot product needs the vectors to be the same size.
- Note: the dot product of two vectors is a number.

Ex: $(1, 2) \cdot (-3, 4) = 5$.

Ex: $(1, 2)$ and $(1, 2, 3)$ cannot be dotted together.

Def: Given an $m \times n$ matrix $A = (a_{ij})$ and an $n \times p$ matrix $B = (b_{k\ell})$, the product of A and B is the $m \times p$ matrix whose (i, j) entry is row i of A dotted with row j of B.

- Note the general rule $(m \times n)(n \times p) = (m \times p)$

Ex:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} -2 & 4 & 1 & 3 \\ 3 & 2 & 0 & 0 \\ 0 & 1 & -2 & 2 \end{pmatrix}$$

Ex:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

Ex:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ 4x + 5y + 6z \end{pmatrix}$$

- Note that the system of equations

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ \vdots &= \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

can now be written as

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

- This will be a very helpful perspective for us soon: Solving the system requires finding all of the x_i, which we've helpfully separated. If we could divide by matrices...
- This is all I can say for now about why we multiply matrices this way.

3.2 Matrix Multiplication Properties

Ex: Let $A = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$ and let $B = \left(\begin{smallmatrix} -1 & 5 \end{smallmatrix}\right)$. Find AB and BA.

- Observation: If AB makes sense, BA doesn't have to.

Ex: Let $A = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$ and let $B = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Compute AB and BA.

- Observation: If AB makes sense, BA can make sense, but it does not have to equal AB.

TPS: If AB and BA both make sense, what can you say about their dimensions?

Def: A matrix is square if it has the same number of rows as columns.

Ex: Let $A = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$ and let $B = \left(\begin{smallmatrix} -4 & 2 \\ 3 & -1 \end{smallmatrix}\right)$. Compute AB and BA.

- Observation: sometimes $AB = BA$.
- This is the one weird property. Everything else works out like you expect. Let A, B, C be matrices, sized so that the following operations make sense. Let $r, s \in \mathbb{R}$.
  - Left distribution: $A(rB + sC) = r(AB) + s(AC)$
  - Right distribution: $(rA + sB)C = r(AC) + s(BC)$
  - Associativity: $(AB)C = A(BC)$.

Ex: Check that $(AB)C = A(BC)$ with matrices $A = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and $C = \left(\begin{smallmatrix} -1 & 8 \\ 3 & -4 \end{smallmatrix}\right)$.

Def: The (main) diagonal of a matrix are all of the entries of the form (i, i).

Ex: The diagonal of (nonsquare matrix here) is ...

Def: The $n \times n$ identity matrix is $I_n = \ldots$.

Ex: Compute $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{smallmatrix}\right)$.

- Observation: if A is $m \times n$, then $I_m A = A = A I_n$.

Ex: Compute $\left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

- Observation: $A0 = 0A = 0$.

Def: The zero matrix is any matrix all of whose entries are 0.

Ex: Let $A = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$ and let $B = \left(\begin{smallmatrix} -4 & 2 \\ 3 & -1 \end{smallmatrix}\right)$. Compute $(AB)^\mathsf{T}$, $A^\mathsf{T} B^\mathsf{T}$, and $B^\mathsf{T} A^\mathsf{T}$.

4 Matrix Inverses

4.1 What They Are and Why They're Useful

- Now that we know how to add, subtract, and multiply matrices, we want to think about how to divide by matrices.
- This seems very useful!
- If we want to solve the linear system

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

  , then it would be very helpful if we could "divide" by A!

- Let's recall some things about numeric division before we do matrix division:
  - Division is a special case of multiplication: dividing by a is the same as multiplying by $1/a$.
  - $1/a$ has a special name: it's called the inverse of a and we can write it as a^{-1}.
  - The inverse of a has a special property: $a^{-1}a = aa^{-1} = 1$.
  - Not every number has an inverse! 0 doesn't have an inverse, for instance.
- Here's our analogy:
- I_n takes the role of 1.
- Not every matrix is going to have an inverse.
- Nonsquare matrices: hopeless.

Def: An $n \times n$ matrix A is invertible if there is a matrix B so that

$$AB = BA = I_n.$$

The matrix B is called the inverse of A and it is denoted A^{-1}.

Ex: Let $A = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$. Observe that

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and vice versa. We can conclude that $A^{-1} = \left(\begin{smallmatrix} -2 & 1 \\ 3/2 & -1/2 \end{smallmatrix}\right)$.

- Fact: We didn't actually need to check both AB and BA. If one of them is the identity, then so is the other.

Ex: Solve the system of equations

$$\begin{aligned} x + 2y &= -2 \\ 3x + 4y &= 1 \end{aligned}$$

- Wow, inverses are great! At least, when they're magically handed down to us from on high...

TPS: True or false? If $A^3 = \left(\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right)$, then A must be invertible.

4.2 How to Find Inverses

Ex: Find the inverse of $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$.

- Start by writing out what this means: we want to find x, y, w, z with

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x & z \\ y & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- This is the same as finding x, y with $A(x, y) = (1, 0)$ and $A(z, w) = (0, 1)$.
- So we solve both of these systems the usual way: write the augmented matrices and put them in RREF.
- Note that we could save time by doing it all at once: put $(A|I_2)$ into RREF.

Thm: (Matrix Inversion Algorithm) Suppose that A is an $n \times n$ matrix. Put the matrix $(A|I_n)$ in RREF, so that it has the form $(R|B)$. If $R = I_n$, then A is invertible and B is the inverse of A. Otherwise, A is not invertible.

Ex: Find the inverse (if it exists) of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 4 \end{smallmatrix}\right)$.

Ex: Find the inverse (if it exists) of $\left(\begin{smallmatrix} 1 & 2 \\ -3 & -6 \end{smallmatrix}\right)$.

- Reminder: check your work!!!
- Note that this means: A is invertible if and only if the RREF of A is the identity matrix.

Ex: Find the value(s) of k that make the following matrix invertible:

$$\begin{pmatrix} 1 & -2 \\ k & 7 \end{pmatrix}$$

4.3 Properties of Inverses

- Here are a few useful properties. Let A and B be invertible, $n \times n$ matrices.
  - $(A^{-1})^{\mathsf{T}} = (A^{\mathsf{T}})^{-1}$.
  - $(AB)^{-1} = B^{-1}A^{-1}$
- Note why $(AB)^{-1}$ is not equal to $A^{-1}B^{-1}$.

Ex: Recall that the inverse of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 4 \end{smallmatrix}\right)$ is equal to $\left(\begin{smallmatrix} 4 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Find the inverse of $\left(\begin{smallmatrix} 0 & 3 \\ -3 & 12 \end{smallmatrix}\right)$.

- Observation: if k is not zero and A is invertible, then $(kA)^{-1} = \frac{1}{k}A^{-1}$.

Ex: Solve the following matrix equation for A:

$$\left(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} A\right)^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

5 Elementary Matrices

5.1 Representing Row Operations As Matrices

- Goal: for non-invertible matrices, we would like to have something kinda close to an inverse.
- We know that if A is a non-invertible matrix, we can't get something like $BA = I_n$.
- But could we get something like $BA = R$ where R is the RREF of A?
- Yes.
- To find this B, we're going to look at encoding Gaussian elimination as a matrix.

Ex: Put the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$$

in RREF. Next, compute

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

- Fact: given an $m \times n$ matrix A, you can swap rows i and j of A by multiplying A on the left by the $m \times m$ identity matrix with rows i and j swapped.

Ex: Put the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

in RREF. Next, compute

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} A$$

- Fact: given an $m \times n$ matrix A, you can multiply row i of A by the constant c if you multiply A on the left by the $m \times m$ identity matrix which has row i multiplied by c.

Ex: Put the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

in RREF. Next, compute

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

- Fact: given an $m \times n$ matrix A, you can add c times row i to row j by multiplying A on the left by the $m \times m$ identity matrix where c times row i was added to row j.

Def: A matrix is an elementary matrix if it was obtained from an identity matrix by applying a single row operation.

Ex: Write down the elementary matrix which will add 5 times row 2 to row 3 of a 4×7 matrix.

5.2 Inverses of Elementary Matrices

- Now that we have matrices for row operations, we will be able to represent sequences of row operations as matrices.
- Before we do that, a quick word on the inverses of elementary matrices.

Ex: What row operation does the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ represent? What is its inverse?

- Observation: elementary matrices which swap two rows are their own inverses.

Ex: What row operation does the matrix $\left(\begin{smallmatrix} 1/3 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ represent? What is its inverse?

- Observation: the inverse of the elementary matrix which multiplies row i by c is the elementary matrix which multiplies row i by $1/c$.

Ex: What row operation does the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 5 & 1 \end{smallmatrix}\right)$ represent? What is its inverse?

- Observation: the inverse of the elementary mtarix which adds c times row i to row j is the elementary matrix which subtracts c times row i to row j.
- Big picture: the inverse of an elementary matrix is an elementary matrix

5.3 Near Inverses

Ex: Let

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix}$$

and let R be the RREF of A. Find a matrix V so that $R = VA$.

- Slow (but reliable) way: Keep track of matrices, then multiply at the end.
- Shortcut (unreliable, unless you can remember exactly what this gets you): Write down augmented matrix $(A|I_2)$ and row reduce to $(R|V)$.
- Note: if you want to find U so that $A = UR$, this process is helpful, but not sufficient.

Ex: Write $\left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$ as a product of elementary matrices.