Chapter 3: Determinants

Greg Knapp

January 22, 2024

1 Basic Techniques and Properties

1.1 Computing Determinants

- Goal: lets find a way to decide when matrices are invertible without actually bothering to invert them.
- The 1×1 case:
  - Consider a 1×1 matrix $A = (a_{11})$.
  - A is invertible if and only if $a_{11} \neq 0$
  - Okay, that's kind of boring. But important to write down.
- The 2×2 case:
  - Consider $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.
  - We'd like to try to put it in RREF.
  - But we have some annoying cases:
  - Case $a \neq 0$: Put the matrix in RREF. Note that invertibility corresponds to $ad - bc \neq 0$
  - Case: $a = 0$, $c \neq 0$: Swap rows 1 and 2, put the matrix in RREF. Note that invertibility corresponds to $ad - bc \neq 0$.
  - Case: $a = 0, c = 0$: A is not invertible (because the RREF can't have a pivot in the first column) and $ad - bc = 0$, so the connection still holds.
- Lesson: A is invertible if and only if some mysterious quantity is nonzero.
- We're going to give that quantity a name and we're going to compute it in the general case.
- That quantity will be called the <u>determinant</u>.
- This definition is going to be recursive, meaning that we will be assuming that you know how to take determinants of "small" matrices and we will be leveraging that to take determinants of "large" matrices.

Def: The determinant of a 1×1 matrix $A = (a_{11})$ is the number $\det(A) = a_{11}$.

Def: The determinant of a 2×2 matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is $\det(A) = ad - bc$.

- Before we learn to compute determinants in general, however, we need a few technical definitions.

Def: Let A be a square matrix. The <u>ijth minor</u> of A is the determinant of the matrix you get by deleting row i and column j from A. It is denoted $\text{minor}(A)_{ij}$.

- Note: the ijth minor is a number!

Ex: Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

The 22 minor of A is...The 13 minor matrix of A is...

- From here, we're going to associate a sign (plus or minus) to each minor.

Def: The ijth cofactor of A is the number

$$\operatorname{cof}(A)_{ij} = (-1)^{i+j} \operatorname{minor}(A)_{ij}.$$

Ex: From before, the 22 cofactor is...the 13 cofactor is...

Def: The determinant of the $n \times n$ matrix $A = (a_{ij})$ is

$$\det(A) = a_{11} \operatorname{cof}(A)_{11} + a_{12} \operatorname{cof}(A)_{12} + \cdots + a_{1n} \operatorname{cof}(A)_{1n}$$

- This is a terrible definition. Let's see it in practice.

Ex: Compute the determinant of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Ex: Compute the determinant of

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & 0 & 3 & -1 \\ 0 & 0 & -3 & -1 \\ 4 & 1 & 0 & 0 \end{pmatrix}$$

Result: $\det(A) = -6$.

- Here's an interesting observation: you can compute the determinant of a matrix by expanding along any row or column.

Thm: Let $A = (a_{ij})$ be an $n \times n$ matrix. Then

$$\begin{aligned} \det(A) &= a_{11} \operatorname{cof}(A)_{11} + a_{12} \operatorname{cof}(A)_{12} + \cdots + a_{1n} \operatorname{cof}(A)_{1n} \\ &= a_{21} \operatorname{cof}(A)_{21} + a_{22} \operatorname{cof}(A)_{22} + \cdots + a_{2n} \operatorname{cof}(A)_{2n} \\ &\vdots \\ &= a_{n1} \operatorname{cof}(A)_{n1} + a_{n2} \operatorname{cof}(A)_{n2} + \cdots + a_{nn} \operatorname{cof}(A)_{nn} \\ &= a_{11} \operatorname{cof}(A)_{11} + a_{21} \operatorname{cof}(A)_{21} + \cdots + a_{n1} \operatorname{cof}(A)_{n1} \\ &\vdots \\ &= a_{1n} \operatorname{cof}(A)_{1n} + a_{2n} \operatorname{cof}(A)_{2n} + \cdots + a_{nn} \operatorname{cof}(A)_{nn} \end{aligned}$$

Def: The process of computing $\det(A)$ by expanding along a row or column is called Laplace expansion.

Ex: Compute the determinant of the matrix

$$\begin{pmatrix} -1 & 4 & 3 & 0 & 0 & 3 \\ -8 & 2 & 0 & 2 & 0 & 1 \\ 3 & -1 & 4 & 2 & 2 & 9 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -9 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & -2 \end{pmatrix}$$

TPS: Explain why a matrix with a row or column of all zeroes must have determinant zero.

- Did we get what we want? Can we tell whether a matrix is invertible based on its determinant?

Thm: A square matrix A is invertible if and only if $\det(A) \neq 0$.

Ex: For which values of a does the following system of linear equations have a unique solution?

$$\begin{aligned} ax + 2y &= -1 \\ x + ay &= 2 \end{aligned}$$

1.2 Determinant Properties

- Goal: compute determinants cleverly.
- Note: this is not how you would program a computer, but rather, how you would compute a determinant by hand as a human.
- The easiest case: diagonal matrices.

Def: A square matrix $A = (a_{ij})$ is <u>upper triangular</u> if $a_{ij} = 0$ whenever $i > j$. A is <u>lower triangular</u> if $a_{ij} = 0$ whenever $i < j$.

- Draw the picture.

Ex: Compute the determinant of

$$A = \begin{pmatrix} 2 & \sqrt{2} & \pi \\ 0 & -1 & e \\ 0 & 0 & 3 \end{pmatrix}$$

- Fact: the determinant of a triangular matrix is the product of its diagonal entries.
- A way to compute determinants is to turn them into triangular matrices using row operations.
- But do row operations affect determinants?
- Row operations correspond to matrix multiplication by the elementary matrices.
- So we need two pieces here: determinants and multiplication, determinants and elementary matrices.
- Fact: If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.
- Fact: the determinants of the elementary matrices are:
  - 1 for adding a scalar multiple of a row to another row
  - c for multiplying a row by c
  - -1 for swapping two rows.

Ex: Compute the determinant of

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 0 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

  - Swap rows 1 and 2
  - Multiply row 1 by 1/2
  - $-3r_1 + r_3$
  - Note that $\det(R) = \det(E_3E_2E_1A)$

- End result:
  - swapping two rows multiplies the determinant by -1
  - adding a scalar multiple of a row to another row doesn't affect the determinant
  - multiplying a row by c multiplies the determinant by c.

TPS: Explain why a matrix with two identical rows must have determinant equal to zero.

- What is $\det(cA)$ for an $n \times n$ matrix A?
- How do determinants interact with other things we can do with matrices, like transposes and inverses?

- Fact: $\det(A^\mathsf{T}) = \det(A)$.
- Fact: If A is invertible, then $\det(A^{-1}) = \det(A)^{-1}$

Ex: Suppose that A and B are 5×5 matrices with $\det(A) = 4$ and $\det(B) = -3$. Compute

$$\det(4AB^{-1}A^{-1}B^3A^\mathsf{T}).$$

- Warning: $\det(A + B) \neq \det(A) + \det(B)$

2 Applications

2.1 Inversion

Ex: Let $A = \left(\begin{smallmatrix} 1 & -2 \\ 3 & 4 \end{smallmatrix}\right)$. What is A^{-1}? What is $\det(A)$? How do they seem related?

Def: For a square matrix A, the <u>cofactor matrix</u> of A is the matrix

$$\operatorname{cof}(A) = (\operatorname{cof}(A)_{ij}).$$

Ex: The cofactor matrix of $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is...

Def: For a square matrix A, the <u>adjugate matrix</u> of A is the matrix $\operatorname{adj}(A) = \operatorname{cof}(A)^\mathsf{T}$.

Ex: The adjugate matrix of $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is...

- Fact: $A \operatorname{adj}(A) = \det(A) I_n$.
- Corollary: if A is invertible, then $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$.

Ex: Compute the inverse of the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & 2 \\ 0 & 0 & -6 \end{pmatrix}$$

2.2 Cramer's Rule

Thm: Suppose that A is an invertible $n \times n$ matrix and B is an $n \times 1$ vector. Let $X = (x_1, \ldots, x_n)^\mathsf{T}$. Let A_i denote the matrix A where column i has been replaced by B. Then the solution to the system of equations $AX = B$ is given by

$$x_i = \frac{\det(A_i)}{\det(A)}.$$

Ex: Solve the system of equations

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 + 2x_2 + x_3 &= 2 \\ 2x_1 - 3x_2 + 2x_3 &= 3 \end{aligned}$$

2.3 Polynomial Interpolation

- This subsection has nothing to do with anything in this chapter, but it's nice to cover at some point.
- Recall: a polynomial is a function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. n is called the degree of $f(x)$.

Ex: Find the degree 2 polynomial which passes through the points $(-2, 1)$, $(1, 2)$, and $(2, 4)$.