Chapter 6: Complex Numbers

Greg Knapp

March 13, 2024

1 Arithmetic

- Fact: $x^2 - 1 = 0$ has two real solutions: ± 1
- The equation $x^2+1=0$ has no real solutions–squaring a real number makes it nonnegative, and adding 1 means you'll never get to zero.
- But that's really inconvenient.
- $x^2 + 1 = 0$ should have two solutions! Just like $x^2 - 1 = 0$.
- So let's make it happen.
- Let's make up a number called i. It's a solution to $x^2 + 1 = 0$.
- This means that $i^2 = -1$.
- What's the other solution?
- $(-i)^2 = i^2 = -1$, so $-i$ is the other solution.
- Now let's make up a system of numbers that has all of our usual real numbers, together with i.
- We'd better be able to multiply i with our usual numbers, so we'll get numbers that look like $5i$, $-2i$, πi, etc.
- We'd also better be able to add multiples of i to our usual numbers, so we'll get numbers like $1 + i$, $3 - 7i$, and $\sqrt{2} + \pi i$.
- This is exactly the set of numbers that we'll look at:

Def: A <u>complex number</u> is any number of the form $a + bi$ where $a, b \in \mathbb{R}$. The set of complex numbers is

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}.$$

Def: For a complex number $a + bi$, a is called the <u>real part</u> and b is called the <u>imaginary part</u>.

Ex:

$$\Re[\sqrt{2} - \pi i] = \sqrt{2} \text{ and } \Im[\sqrt{2} - \pi i] = -\pi.$$

- Note: The imaginary part is a real number! It's called the imaginary part because it is the (real) number that is multiplied by i.
- I claim that we can add any two complex numbers together, and we'll get another complex number:

$$(a + bi) + (x + yi) = (a + x) + (b + y)i.$$

- E.g.

$$(7 + 2i) + (-1 + 3i) = 6 + 5i.$$

- I claim that we can multiply any complex numbers together and we'll get another complex number:

$$(a+bi)(x+yi) = ax + ayi + bxi + byi^2 = (ax - by) + (ay + bx)i.$$

Ex:

$$(7+2i)(-1+3i) = -7 + 6i - 2i + 6i^2 = -13 + 4i.$$

- I claim that we can even divide complex numbers together, though this is not obvious, and requires an intermediate tool.

Def: The <u>complex conjugate</u> of a complex number $a+bi$ is the number $a-bi$. We use the notation

$$\overline{a+bi} = a - bi.$$

The notation $\overline{w}$ is read as "w bar."

Ex: The complex conjugate of $\sqrt{2} + \pi i$ is $\sqrt{2} - \pi i$.

Ex: The complex conjugate of $1 - 2i$ is $1 + 2i$.

Ex: The complex conjugate of $3i = 0 + 3i$ is $0 - 3i = -3i$.

Ex: The complex conjugate of $7 = 7 + 0i$ is $7 - 0i = 7$.

- Now to division:

$$\frac{a+bi}{x+yi} = \frac{a+bi}{x+yi} \cdot \frac{x-yi}{x-yi} = \frac{ax - ayi + bxi - byi^2}{x^2+y^2} = \frac{(ax+by)+(bx-ay)i}{x^2+y^2} = \frac{ax+by}{x^2+y^2} + i \cdot \frac{bx-ay}{x^2+y^2}.$$

Ex:

$$\frac{7+2i}{-1+3i}.$$

- Since all of our usual arithmetic properties hold, we can solve many equations just like you expect:

Ex: Find the $z \in \mathbb{C}$ so that

$$\frac{z+i}{z+1} = 3 - 2i.$$

- We can also solve equations which involve the complex conjugation operation, though they may take a little more effort.

- To see this, we need a few facts about complex conjugation:
  - $\overline{z+w} = \overline{z} + \overline{w}$.
  - $\overline{zw} = \overline{z}\overline{w}$.
  - $\overline{(\overline{z})} = z$
  - $\overline{z/w} = \overline{z}/\overline{w}$.

Ex: Find $z \in \mathbb{C}$ so that $\overline{z+i} = 3 - 2i + 3z$.

- Equivalent to $\overline{z} - i = 3 - 2i + 3z$.
- I.e. $\overline{z} - 3z = 3 - i$.
- Now write $z = a + bi$, so we get $a - bi - 3(a+bi) = 3 - i$.
- Hence, $-2a = 3$ and $-4b = -1$, i.e. $a = -2/3$ and $b = 1/4$.

2 Complex Numbers as Vectors

- There are some similarities between complex numbers and vectors.
- Note that a complex number is defined by two real numbers.
- So maybe we should look for similarities between $\mathbb{C}$ and $\mathbb{R}^2$.
- Let's compare addition operations:
$$(a+bi)+(x+yi)=(a+x)+(b+y)i \text{ and } \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a+x \\ b+y \end{pmatrix}.$$
- Wow, those are pretty similar. What about scalar multiplication (by some $c \in \mathbb{R}$)?
- $$c(a+bi)=ca+cbi \text{ and } c\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ca \\ cb \end{pmatrix}.$$
- Again, this looks exactly the same.
- They are similar enough that we can visualize $\mathbb{C}$ in the same way we visualize $\mathbb{R}^2$ and everything we know from $\mathbb{R}^2$ will carry over to $\mathbb{C}$.
- We can draw the number $a+bi$ as the point (a,b).
- Draw some points.
- Adding complex numbers is exactly like adding vectors.
- We can port over some concepts from $\mathbb{R}^2$ to $\mathbb{C}$, like length.

Def: The modulus/absolute value of the number $a+bi$ is $|a+bi|=\sqrt{a^2+b^2}$.

- Recall the triangle inequality from $\mathbb{R}^2$: $\|\vec{v}+\vec{w}\| \leqslant \|\vec{v}\|+\|\vec{w}\|$.
- The same is true for $\mathbb{C}$: $|z+w| \leqslant |z|+|w|$.
- Here's an interesting fact:
$$(a+bi)(\overline{a+bi})=a^2+b^2=|a+bi|^2.$$
- A shorter way of writing this is $z\overline{z}=|z|^2$.
- Another concept we can bring from $\mathbb{R}^2$ is that of angles.
- Rather than thinking about angles between vectors, we're going to focus on the angle a vector makes with the positive real/horizontal axis.

Q: If I have a complex number with absolute value r and which makes an angle θ with the positive real axis, how do I express that in the form $a+bi$?

- Draw the triangle, $a=r\cos\theta$ and $b=r\sin\theta$.
- So the complex number with magnitude r and which makes angle θ with the positive real axis is $r\cos\theta+(r\sin\theta)i=r(\cos\theta+i\sin\theta)$.

Def: The polar form of a complex number is $z=r(\cos\theta+i\sin\theta)$. θ is called the argument. If $-\pi<\theta\leqslant\pi$, then θ is called the principal argument.

Ex: What is the polar form of $-1-i$? What are the possible arguments of $-1-i$? What is the principal argument of $-1-i$?

3 Complex Numbers as Numbers

- Complex numbers are numbers, so we should be able to do usual number things with them!
- Let's talk about exponents.
- For any $z \in \mathbb{C}$, we can easily talk about z^n for any integer n.
- If n is positive, then $z^n = z \cdot z \cdots z$.
- If n is negative, then $z^n = 1/(z \cdot z \cdots z)$.
- What about $z^{1.5}$? This is much harder and we'll come back to it.
- Another thing we can do is look at b^z where b is a positive real number and any complex number.
- How could we possibly make sense of this?
- Let's start with our favorite positive real number, e.
- We want to think about what e^{x+iy} should be.
- Part of that answer is easy: $e^{x+iy} = e^x \cdot e^{iy}$ and we know what e^x should be.
- e^{iy} is trickier though.
- If you've taken calculus, you might have seen the following three things:

$$\begin{aligned} e^y &= 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots + \frac{y^n}{n!} + \dots \\ \cos(y) &= 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots + \frac{(-1)^n y^{2n}}{(2n)!} + \dots \\ \sin(y) &= y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots + \frac{(-1)^n y^{2n+1}}{(2n+1)!} + \dots \end{aligned}$$

  and you probably said, "wow, those things sure look really similar."

- It turns out they are: let's look at e^{iy}.
- Woah, it's $e^{iy} = \cos(y) + i\sin(y)$.
- And this is how we make sense of e^{iy} as a complex number.
- Now recall the polar form of a complex number: $r(\cos\theta + i\sin\theta)$
- We can rewrite this more simply as $re^{i\theta}$.

Ex: What is the polar form of -7? $-7 = 7e^{i\pi}$.

Ex: Compute $(1+i)^2$ in two different ways.

- $(1+i)^2 = 1 + 2i - 1 = 2i$.
- $(1+i)^2 = (\sqrt{2}e^{i\pi/4})^2 = 2e^{i\pi/2} = 2(\cos(\pi/2) + i\sin(\pi/2)) = 2i$.

- Wow, the second way seems worse!

Ex: Compute $(1+i)^{100}$ in two different ways.

- $(1+i)^{100} = \dots$ uh oh.
-
$$(1+i)^{100} = (\sqrt{2}e^{i\pi/4})^{100} = 2^{50}e^{25i\pi} = 2^{50}(\cos(25\pi) + i\sin(25\pi)) = -2^{50}.$$

- Now that we've seen that polar form makes z^n easier for integers n, let's return to $z^{1.5}$.
- $z^{1.5} = z^{3/2}$, so we had better learn some things about fractional powers.
- Let's start with fractions that look like $z^{1/n}$ for a positive integer n.
- Recall that this means "nth root."
- Problem: sometimes there are multiple nth roots.
- 1 has two square roots: ± 1.
- -1 has two square roots: $\pm i$.
- Let's start with square roots: these are "easy enough" to do with brute force:

Ex: Find every square root of $7 + 24i$.

- We can do this by solving the equation $z^2 = 7 + 24i$.
- To do this, we can write $z = a + bi$ and try to solve for a and b.

$$\begin{aligned} (a+bi)^2 &= 7 + 24i \\ \Rightarrow a^2 - b^2 + 2abi &= 7 + 24i. \end{aligned}$$

- From here, set the real and imaginary parts equal to one another:
- $a^2 - b^2 = 7$ and $2ab = 24$.
- Use the second equation to solve for b: $b = 12/a$.
- Substitute into the first equation:

$$a^2 - \left(\frac{12}{a}\right)^2 = 7 \Rightarrow a^4 - 144 = 7a^2 \Rightarrow a^4 - 7a^2 - 144 = 0.$$

- Now solve the quadratic in a^2. Can substitute $u = a^2$ if you like.

$$a^2 = \frac{7 \pm \sqrt{49 + 4 \cdot 144}}{2} = -9, 16.$$

- Recall that we want real numbers a which work, so $a = \pm 4$.
- Now $b = 12/a$ yields that the roots of $7 + 24i$ are $4 + 3i$ and $-4 - 3i$.

- This will always work for square roots.
- For cube roots on the other hand, this doesn't work so well...

Ex: Find every cube root of $1 + i$.

- We're looking for numbers z so that $z^3 = 1 + i$.
- Polar form is extremely helpful here.
- $z = re^{i\theta}$.
- $1 + i = \sqrt{2}e^{i\pi/4}$.
- We want to find r and θ so that $r^3 e^{3i\theta} = \sqrt{2}e^{i\pi/4}$.
- So we'd better have: $r^3 = \sqrt{2} = 2^{1/2}$ and $e^{3i\theta} = e^{i\pi/4}$.
- The first one is easy to solve: $r = 2^{1/6}$.
- The second one is trickier than you think.
- The "obvious" answer is $\theta = \pi/12$. You get this by solving $3\theta = \pi/4$.

- But there are two less obvious answers as well: $\theta = 3\pi/4$ and $\theta = 17\pi/12$.
- Why do these work? For $\theta = 3\pi/4$:

$$e^{3i\theta} = e^{9i\pi/4} = \cos(9\pi/4) + i\sin(9\pi/4) = \cos(\pi/4) + i\sin(\pi/4) = e^{i\pi/4}.$$

- For $\theta = 17\pi/12$:

$$e^{3i\theta} = e^{17i\pi/4} = e^{i\pi/4}e^{2i\pi} = e^{i\pi/4}.$$

- Okay, it's great to see how they work after the fact, but where did they come from?
- Our three θ values of $\pi/12$, $9\pi/12$, and $17\pi/12$ came from solving:

$$\begin{aligned}3\theta &= \pi/4\\ 3\theta &= \pi/4 + 2\pi\\ 3\theta &= \pi/4 + 4\pi.\end{aligned}$$

- In the end, we have three cube roots: $2^{1/6}e^{i\pi/12}, 2^{1/6}e^{9i\pi/12}, 2^{1/6}e^{17i\pi/12}$.

* In general, there will be n nth roots of any complex number.
* To find the solutions to $z^n = w$, you can:
  - Write the components in polar form $z = re^{i\theta}$ and $w = se^{i\varphi}$.
  - Rewrite the equation $z^n = w$ becomes $r^n e^{in\theta} = se^{i\varphi}$.
  - Solve $r^n = s$.
  - Solve the n different equations:

$$\begin{aligned}n\theta &= \varphi\\ n\theta &= \varphi + 2\pi\\ &\vdots\\ n\theta &= \varphi + (n-1)2\pi\end{aligned}$$

Ex: Find the fourth roots of $-i$.

- Write $z = re^{i\theta}$
- Write $-i = e^{3\pi/2}$.
- Solve $r^4 e^{4i\theta} = e^{3\pi/2}$.
- So $r = 1$.
- To get our θ values: solve

$$\begin{aligned}4\theta &= 3\pi/2\\ 4\theta &= 3\pi/2 + 2\pi\\ 4\theta &= 3\pi/2 + 4\pi\\ 4\theta &= 3\pi/2 + 6\pi\end{aligned}$$

- Now we have $\theta = 3\pi/8, 7\pi/8, 11\pi/8, 15\pi/8$.
- Hence, our fourth roots of $-i$ are $e^{3i\pi/8}, e^{7i\pi/8}, e^{11i\pi/8}$, and $e^{15i\pi/8}$.

4 The Quadratic Formula

- It turns out that the quadratic formula works just as well for quadratics with complex coefficients as for quadratics with real coefficients.
- I want to update our understanding though, since there is a detail that is a little confusing.
- The solutions to
$$az^2 + bz + c = 0$$
are
$$z = \frac{-b + \text{ the two square roots of } b^2 - 4ac}{2a}.$$

Ex: Solve the equation $z^2 + (3+2i)z + 5 + i = 0$.

- Quadratic formula says:
$$z = \frac{-(3+2i) + \text{ the two square roots of } (3+2i)^2 - 4(5+i)}{2}.$$
- So we'd better find the two square roots of
$$(3+2i)^2 - 4(5+i) = (5+12i) - 20 - 4i = -15 + 8i.$$
- Solve
$$(a+bi)^2 = -15 + 8i.$$
- Get $a^2 - b^2 = -15$ and $2ab = 8$.
- Hence, $b = 4/a$.
- So
$$a^2 - \frac{16}{a^2} = -15 \Rightarrow a^4 + 15a^2 - 16 = 0 \Rightarrow (a^2+16)(a^2-1) = 0.$$
- Hence, $a = \pm 1$.
- So our square roots are $1 + 4i$ and $-1 - 4i$.
- So our values of z are:
$$z = \frac{-3 - 2i + 1 + 4i}{2}, \frac{-3 - 2i + (-1 - 4i)}{2}.$$