Upper Bounds on Polynomial Root Separation

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A Thue Equation

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2. Only two solutions (x, y) have $y = 0$—namely, $(\pm 1, 0)$—so assume $y \geqslant 1$.

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3. Only two solutions (x, y) have $y = 1$—namely, $(0, 1)$ and $(-3, 1)$—so assume $y \geqslant 2$.

A Thue Equation: Initial Approach

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Next, factor the left-hand side:

$$\prod_{i=1}^{6} \left| \frac{x}{y} - \alpha_i \right| = \frac{1}{y^6}$$

where $\alpha_1, \ldots, \alpha_6$ are the roots of $X^6 + 3X^5 - 1$.

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Helpful fact?

If (x, y) is a solution with $y \geqslant 2$ and $\alpha_1, \ldots, \alpha_6$ are the roots of $f(X) = X^6 + 3X^5 - 1$, then

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Obstacle (Dirichlet)

For any irrational $\alpha \in \mathbb{R}$, there are infinitely many relatively prime $p, q \in \mathbb{Z}$ with

$$\left|\frac{p}{q} - \alpha\right| \leqslant \frac{1}{q^2}.$$

A Thue Equation: Searching for Improvements

An Inefficiency

Note that

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Roots

Complex roots of $X^6 + 3X^5 - 1$:

$$\alpha_1 \approx -3.004$$
$$\alpha_2 \approx 0.767$$
$$\alpha_3, \alpha_4 \approx -0.66 \pm 0.52i$$
$$\alpha_5, \alpha_6 \approx 0.28 \pm 0.74i$$

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&\geqslant \left| \frac{x}{y} - \alpha_j \right| \prod_{i \neq j} \left| |\alpha_j - \alpha_i| - \left| \frac{x}{y} - \alpha_j \right| \right|
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&\geqslant \left| \frac{x}{y} - \alpha_j \right| \prod_{i \neq j} 0.43 |\alpha_j - \alpha_i| \\
&\geqslant \frac{1}{120} \left| \frac{x}{y} - \alpha_j \right|.
\end{aligned}$$

A New Approximation Result

Before and After

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Conclusion

Using Thue's improvement to Liouville's Theorem, there are only finitely many such $\frac{x}{y}$.

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Using Thue's improvement to Liouville's Theorem, there are only finitely many such $\frac{x}{y}$.

Key Ingredient

We used a lower bound on $|\alpha_j - \alpha_i|$.

Relevant Definitions

Let

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$$M(f) := |b_n| \prod_{1 \leqslant j \leqslant n} \max(1, |\alpha_j|).$$

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$$M(f) := |b_n| \prod_{1 \leqslant j \leqslant n} \max(1, |\alpha_j|).$$

- The (absolute value of the) discriminant of $f(x)$ is

$$|\Delta_f| := |b_n|^{2n-2} \prod_{1 \leqslant i < j \leqslant n} |\alpha_i - \alpha_j|^2.$$

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Meaning

$M(f)$ translates between roots and coefficients.

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Goal

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Theorem (Mahler, 1964)

For all monic polynomials $f(x) \in \mathbb{C}[x]$ *of degree* $n \geqslant 2$,

$$\operatorname{sep}(f) \geqslant \frac{\sqrt{3|\Delta_f|}}{n^{(n+2)/2} M(f)^{n-1}}.$$

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Corollary (Mahler, 1964)

For separable monic $f(x) \in \mathbb{Z}[x]$ of degree $n \geqslant 2$,

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If $n \geqslant 4$ we get the nicer expression

$$\operatorname{sep}(f) < n^{\frac{1}{n-3}} M(f)^{\frac{2}{n-\frac{1}{2}}}.$$

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Question

Is this an optimal upper bound on $\operatorname{sep}(f)$?

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Procedure

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- If we sampled random polynomials in $\mathbb{Z}[x]$ by randomly choosing their coefficients, we would have to factor them to compute the Mahler measure and separation.
- Instead, to create a "random" polynomial of degree n, we choose its roots from a uniform distribution on an appropriate region of $\mathbb{C}$.

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- We can illustrate the difference between these cases as follows.

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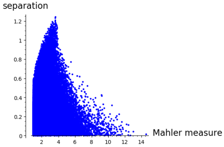
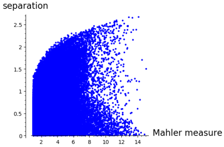
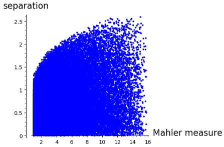


Signature $(4, 0)$



Signature $(2, 1)$



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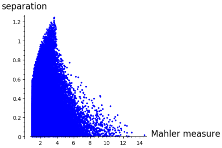

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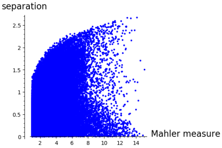
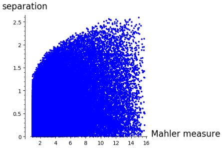


Signature $(2,1)$



Signature $(0,2)$

We're interested in an upper bound of the form $\operatorname{sep}(f) \leqslant C(n)M(f)^{e(n)}$, so a log-log plot makes more sense.

Log-Log Data for Quartics

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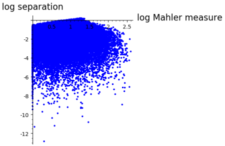

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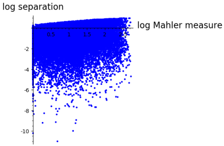
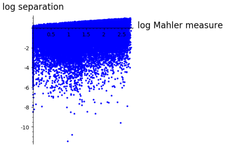


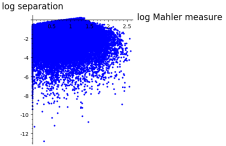
Signature $(2, 1)$



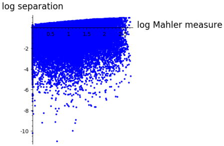
Signature $(0, 2)$

Log-Log Data for Quartics

Here are the results of selecting 50,000 random polynomials with real coefficients of a specified signature, and plotting their logarithmic Mahler measure against their logarithmic separation:



Signature $(4, 0)$



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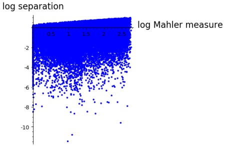
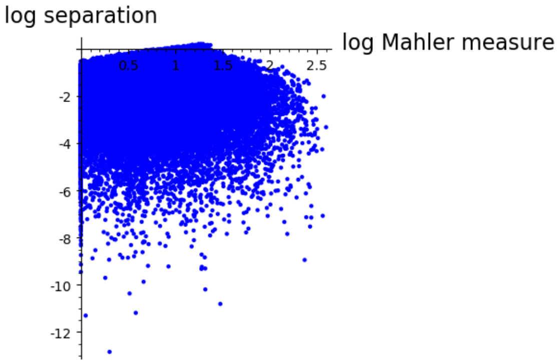


Signature $(0, 2)$

The logarithmic separation appears to be bounded above by a linear function of the logarithmic Mahler measure.

Discerning the Upper Bound: Signature $(4, 0)$



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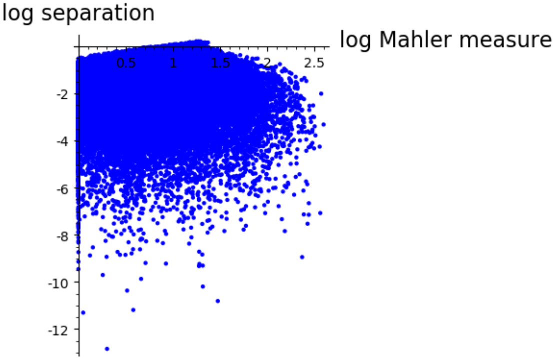

The upper bound here appears to be something like $\log \operatorname{sep}(f) \leqslant \frac{1}{3} \log M(f) - \frac{1}{2}$, i.e.

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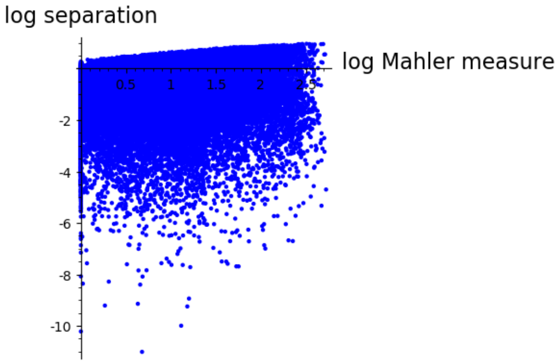

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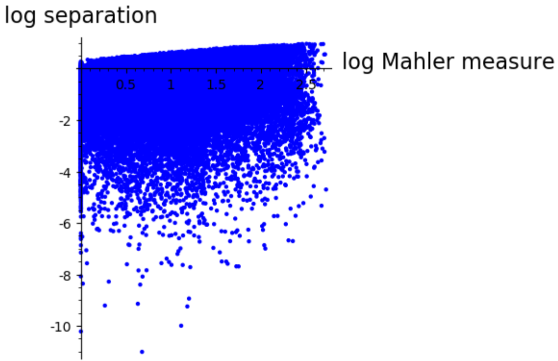

The upper bound in this case appears to be something like $\log \operatorname{sep}(f) \leqslant \frac{1}{3} \log M(f) + \frac{1}{4}$, i.e.

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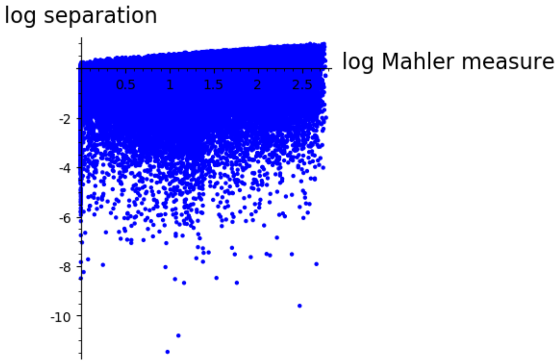
Discerning the Upper Bounds: Signature $(0, 2)$



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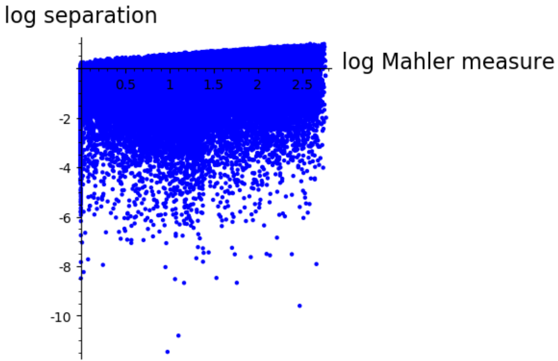

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Here is a plot of the logarithmic separations and Mahler measures of 2,000,000 degree 10 monic polynomials with 5 pairs of complex conjugate roots:

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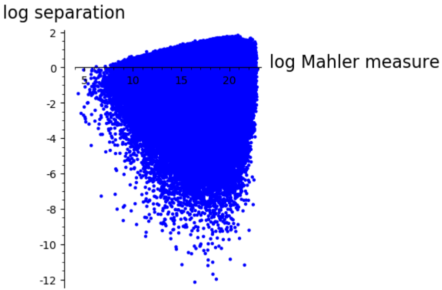

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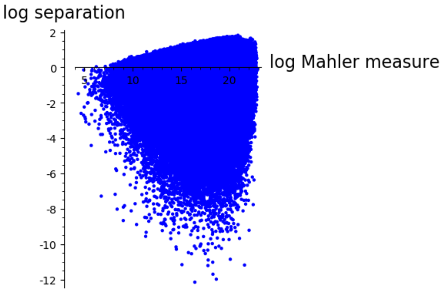

Here, we get something like $\log \operatorname{sep}(f) \leqslant \frac{1}{10} \log M(f) - \frac{1}{2}$, i.e.

$$\operatorname{sep}(f) \leqslant e^{-1/2} M(f)^{1/10}.$$

A Conjecture

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Write $f(x) = \prod_{i=1}^{n}(x - \alpha_i)$ so that $|\alpha_1| \leqslant |\alpha_j|$ for all j.

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Question

Is this an optimal upper bound on $\operatorname{sep}(f)$?

Main Theorem

Theorem (K., Yip, 2025)

Let $f(x) \in \mathbb{C}[x]$ be monic and separable of degree $n \geqslant 2$. Then

$$\operatorname{sep}(f) \leqslant \min\left\{2, \frac{34}{\sqrt{n}}\right\} M(f)^{1/(n-1)}.$$

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Note

This result is sharp except possibly for the constant 34. The best possible constant is at least $5/8$.

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Rearranging yields the theorem.

Big Picture

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What do we gain by knowing $\mathrm{sep}(f) \leqslant \frac{34M(f)^{1/(n-1)}}{\sqrt{n}}$?

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For every monic, noncyclotomic, irreducible $f(x) \in \mathbb{Z}[x]$, $M(f) \geqslant 1.1$.

Consequence

Lehmer's conjecture holds for $f(x)$ with

$$\operatorname{sep}(f) \geqslant \frac{34 \cdot (1.1)^{1/(n-1)}}{\sqrt{n}}.$$

The Other Side

Another Consequence

From Mahler's earlier $\operatorname{sep}(f) \geqslant \frac{\sqrt{3}}{n^{(n+2)/2} M(f)^{n-1}}$, it follows that Lehmer's conjecture holds when

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A Reduction

To prove Lehmer's conjecture, it suffices to assume

$$\frac{\sqrt{3}}{n^{(n+2)/2} \cdot 1.1^{n-1}} < \operatorname{sep}(f) < \frac{34 \cdot (1.1)^{1/(n-1)}}{\sqrt{n}}.$$

Thank you!

Questions?