Bounds on the Number of Solutions to Thue Equations

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Land Acknowledgment

The University of Calgary, located in the heart of Southern Alberta, both acknowledges and pays tribute to the traditional territories of the peoples of Treaty 7, which include the Blackfoot Confederacy (comprised of the Siksika, the Piikani, and the Kainai First Nations), the Tsuut'ina First Nation, and the Stoney Nakoda (including Chiniki, Bearspaw, and Goodstoney First Nations). The City of Calgary is also home to the Métis Nation of Alberta Region 3.

Intro to Thue Equations

Definition

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Let $F(x, y)$ be an integral binary form which is irreducible over $\mathbb{Z}$ and has degree at least 3. Let h be an integer. Then the equation

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is known as a *Thue equation* and the inequality

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Corollary (Thue, 1909)

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Why all the hypotheses?

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$|F(x, y)| \leqslant h$ has finitely many integer solutions when $F(x, y) \in \mathbb{Z}[x, y]$ has $\deg(F) \geqslant 3$, is irreducible, and is homogeneous.

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Necessity of Hypotheses

- $\deg(F) \geqslant 3$ is necessary: $F(x,y) = x^2 - 2y^2$ is irreducible and homogeneous, and $|F(x,y)| \leqslant 1$ has infinitely many integer-pair solutions.

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- $F(x,y)$ being irreducible is also necessary: if $F(x,y)$ has a linear factor, say $mx - ny$, then any integer multiple of (n,m) is a solution to $|F(x,y)| \leqslant h$.
- The homogeneity condition is also necessary: if $F(x,y) = x^6 + y^3$, then any integer pair of the form $(n, -n^2)$ will be a solution to $|F(x,y)| \leqslant h$.

Follow-up Questions

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- What are the (integer) solutions to $|F(x, y)| \leqslant h$?
- How many solutions are there to $|F(x, y)| \leqslant h$?
- On which features of $F(x, y)$ and h do the number of solutions depend?

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Therefore, for some i,

$$\left| \frac{p}{q} - \alpha_i \right| = \text{small}.$$

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Important Connection

Every solution (p, q) to the Thue inequality $|F(x, y)| \leqslant h$ with $q \neq 0$ yields a good rational approximation $\frac{p}{q}$ to a root α of $f(X) = F(X, 1)$.

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There are many tools (some of which are effective!) to find/count good rational approximations of algebraic numbers.

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- The map from solutions (p, q) with $q \neq 0$ to rational approximations $\frac{p}{q}$ is not injective.
- The map from primitive solutions (p, q) with $q \neq 0$ and $\gcd(p, q) = 1$ to rational approximations $\frac{p}{q}$ is injective.

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- Fact: to count every solution, it suffices to count primitive solutions.

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  - $H = 10$
- Let $h \in \mathbb{Z}_{>0}$

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- Recall that if $F(p, q) = h$, then $\frac{p}{q}$ is close to a root of $f(X) := F(X, 1)$.
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Lemma (Descartes, 1637)

If $g(x) \in \mathbb{R}[x]$ has $s + 1$ nonzero summands, then $g(x)$ has no more than $2s + 1$ real roots.

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Goal

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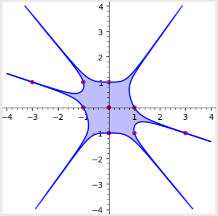
Next Up

What do we expect $N(F,h)$ to look like?

Geometric View of $|F(x,y)| \leqslant h$

A Picture

$|F(x,y)| \leqslant h$ corresponds to a region of the xy-plane:



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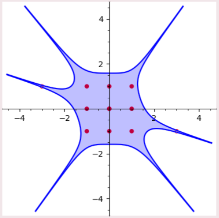

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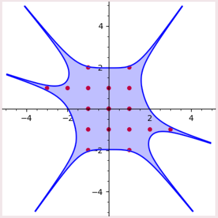

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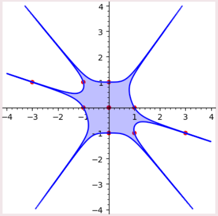

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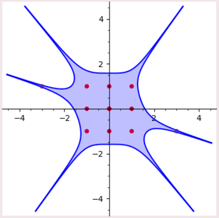

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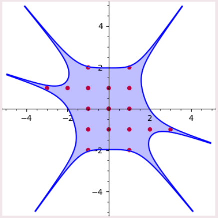


$|x^5 + 3x^4y - y^5| \leqslant 30$

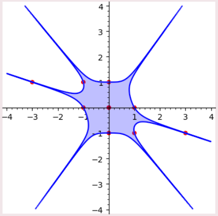
Computing $N(F,h)$

Some values of $N(F,h)$ for $F(x,y) = x^5 + 3x^4y - y^5$:

Geometric View of $|F(x,y)| \leqslant h$

A Picture

$|F(x,y)| \leqslant h$ corresponds to a region of the xy-plane:



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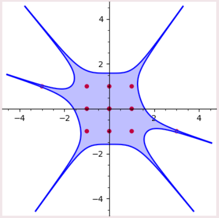

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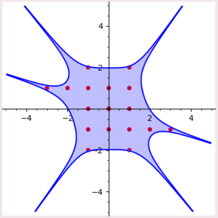

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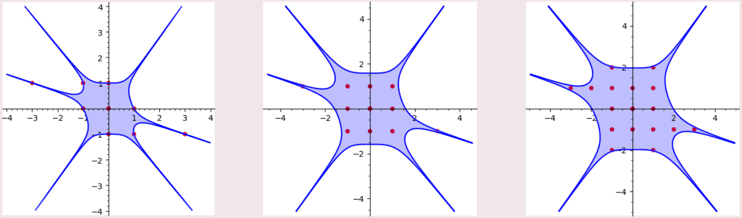
h	1	10	30
$N(F,h)$	9	11	17

Exploring Dependence on h

A Picture



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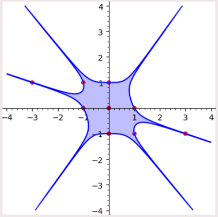

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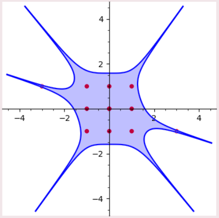

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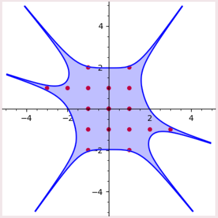

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$$N(F, h) = \text{number of lattice points “inside” } |F(x, y)| \leqslant h$$

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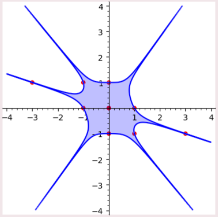

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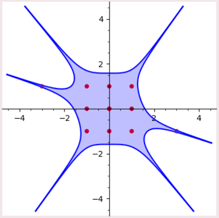

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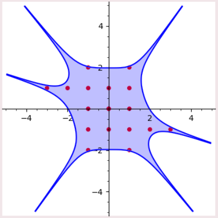

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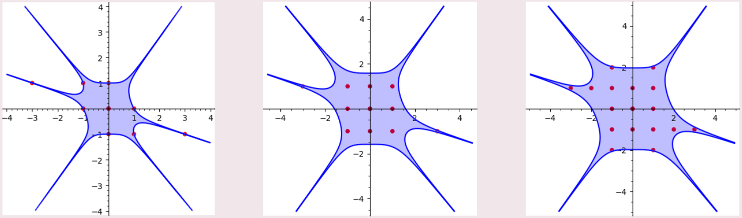

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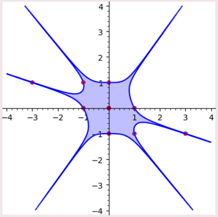

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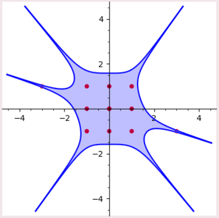

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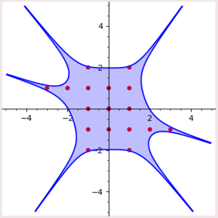

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Exploring dependence on h

Theorem (Mahler, 1934)

Let

$$V(F,1) := \operatorname{vol}\{(x,y) \in \mathbb{R}^2 : |F(x,y)| \leqslant 1\}.$$

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Moral

The factor of $h^{2/n}$ is necessary and sufficient and we expect

$$N(F,h) \approx h^{2/n} \cdot N(F,1).$$

Exploring $N(F, 1)$

Previous Facts

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Number of Approximations per Root

Based on some approximation theorems, we expect the number of rational approximations per root to be absolutely bounded.

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Conclusion

We expect there to be no more than a constant times s solutions to $|F(x, y)| \leqslant 1$.

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Meaning

The symbol $\ll$ means "(is) no more than a constant times."

Conclusion (rephrased)

We expect there to be $\ll s$ solutions to $|F(x,y)| \leqslant 1$.

A Conjecture and Theorem of Mueller and Schmidt

The Pieces

Recall that we expect:

$$N(F,h) \approx h^{2/n} \cdot N(F,1)$$
$$N(F,1) \ll s$$

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Picking Values for s and h

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Theorem (Thomas, 2000)

For $n \geqslant 39$ and $F(x, y) = ax^n + bx^k y^{n-k} + cy^n$, the number of solutions to $|F(x, y)| = 1$ is less than or equal to 48.

Types of Solutions

Separating Solutions

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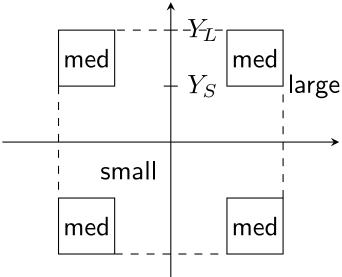

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- Medium solutions to $|F(x, y)| \leqslant h$ produce good approximations to some small set of algebraic numbers (Mueller and Schmidt, 1987).
- Fix one of those algebraic numbers, α, and note that the good rational approximations of α must be close to each other.
- Manipulate inequalities to find that the denominators of the rational approximations must be exponentially far apart.

Counting Medium Solutions

Our improvement involves efficiently bounding t in the situation where both

$$Y_S \leqslant q_0 < q_1 < \cdots < q_t \leqslant Y_L$$

and the inductive relation

$$q_{i+1} > \frac{q_i^{\frac{n}{s}-1}}{K}$$

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Lemma (K., 2021)

If $n \geqslant 3s$ and there are $t+1$ primitive medium solutions associated to α, then

$$t \leqslant \frac{\log\left[\frac{\log Y_L K^{-1/(\frac{n}{s}-2)}}{\log Y_S K^{-1/(\frac{n}{s}-2)}}\right]}{\log\left(\frac{n}{s}-1\right)}.$$

Moreover, this bound is as sharp as possible, given existing approximation results.

Counting Solutions

Theorem (K., 2021)

The number of primitive medium solutions to $|F(x,y)| \leqslant h$ *when* $n \geqslant 3s$ *is*

$$\ll s\left(1+\log\left(s+\frac{\log h}{\max(1,\log H)}\right)\right)$$

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Note: Φ measures the "sparsity" of F and satisfies $\log^3 s \leqslant e^{\Phi} \ll s$.

Asymptotic Bounds

As a consequence:

Theorem (K., 2023)

When $n > 4se^{2\Phi}$, the number of primitive solutions to $|F(x,y)| \leqslant h$ is

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Compare to:

Theorem (Mueller and Schmidt, 1987)

The number of integer pair solutions to $|F(x, y)| \leqslant h$ is

$$\ll s^2h^{2/n}(1 + \log h^{1/n}).$$

Explicit Bounds for Trinomials

Theorem (Thomas, 2000)

If $F(x, y) = ax^n + bx^k y^{n-k} + cy^n$, there are no more than $C_1(n)$ solutions to $|F(x, y)| = 1$ where $C_1(n)$ is defined by

n	6	7	8	9	10-11	12-16	17-37	$\geqslant 38$
$C_1(n)$	136	86	96	62	72	60	56	48

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The above theorem is still true with $C_1(n)$ replaced by $C_2(n)$:

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Question

Is this a good bound?

Trinomial Computations

H	1	2	3	4	5	6	7	8	9	10	$\cdots$	16
$n=6$	8	6	8	8	6	6	6	6	8	6	$\cdots$	12
$n=7$	8	6	8	8	6	6	6	6	8	6	$\cdots$	8
$n=8$	8	6	8	8	6	6	6	6	8	6	$\cdots$	12
$n=9$	8	6	8	8	6	6	6	6	8	6	$\cdots$	8
$n=10$	8	6	8	8	6	6	6	6	8	-	$\cdots$	8
$n=11$	8	6	8	8	6	6	6	6	8	-	$\cdots$	-
$n=12$	8	6	8	8	6	6	6	-	-	-	$\cdots$	-
$n=13$	8	6	8	8	6	6	-	-	-	-	$\cdots$	-
$n=14$	8	6	8	8	6	6	-	-	-	-	$\cdots$	-
$n=15$	8	6	8	8	6	-	-	-	-	-	$\cdots$	-
$n=16$	8	6	8	8	6	-	-	-	-	-	$\cdots$	-
$n=17$	8	6	8	8	-	-	-	-	-	-	$\cdots$	-

Maximum number of solutions to $|F(x,y)|=1$ for any trinomial of height H and degree n

Thank you!

Questions?