# Chapter 2 Lecture Notes

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Section 2.1: Sets and Set Operations

- When working with arguments in chapter 1, we talked a lot about "all men being mortal" and we represented this with a Venn diagram where the set of men was sitting inside the set of mortals.
- The goal of this chapter is to more clearly define what sets are and to be able to count the number of things in a given set. This is going to allow us to later ask questions like "what is the probability of being dealt a full house from a deck of cards?"
- Fair warning, the stuff we're about to do is a ton of definitions and new notation.
- **Def:** A *set* is a collection of objects. Each of the objects is called an *element* or *member* of the set.
- How do we write sets?
- One option is to use roster notation. Roster notation says that you can list a set by listing all of its elements.
- **Ex:** Consider the following set in roster notation: $A = \{1, 2, 3, 4, 5\}$.
- **Ex:** Consider the following set in roster notation: $B = \{\text{Frank Sinatra}, \text{Dean Martin}, \text{Sammy Davis Jr.}\}$
- Alternatively, we can use set-builder notation. Set-builder notation says that you can describe a set as the set of all things with a certain property
- **Ex:** Our set $A$ above can be defined as $A = \{x \mid x \text{ is a whole number and } 1 \leq x \text{ and } x \leq 5\}$
- **Ex:** If we wanted to talk about the set of all books by JK Rowling, it would be very difficult to write the whole set out in roster notation. Instead, we might write $C = \{x \mid x \text{ is a book written by JK Rowling}\}$
- In order to talk about the elements of a set, we use the symbol $\in$. For a set $X$, we say that $y \in X$ if $y$ is an element of $x$. In order to say that $y$ is not an element of $X$, we would say $y \notin X$.
- **Ex:** $1 \in A$. $2.5 \notin A$. *Harry Potter and the Sorcerer's Stone* $\in C$. *The Fellowship of the Ring* $\notin C$. Dean Martin $\in B$. Billy Joel $\notin B$.
- Something good to keep in mind is that the statements "$1 \in A$" and "Billy Joel $\notin B$" are statements (as in the sense of chapter 1). They are either true or false.
- **Def:** One of the sets that is sometimes helpful to talk about is the *empty set*. This is the (unique) set with no elements. It is denoted $\emptyset$.
- One of our driving goals, however, is to be able to count things in sets, so we need some notation for that.
- **Def:** Given a set, $X$. The *cardinal number* (or *cardinality*) of $X$ is the number of elements of $X$. It is denoted $n(X)$.
- **Ex:** The cardinality of $A$ is $n(A) = 5$. The cardinality of $B$ is $n(B) = 3$. The cardinality of $C$ is $n(C) = 21$. The cardinality of $\emptyset$ is $n(\emptyset) = 0$.

- Something that seems like it's barely worth saying (but it is worth saying) is that two sets are the same or equal when they have the same elements. *Order of listing doesn't matter.* The sets $\{2,3,4\}$ and $\{4,2,3\}$ are the same set.
- When we want to visually represent sets and their relations, we do so with a Venn Diagram.
- A Venn Diagram consists of at least two things. First, a rectangle, representing the *universal set* (to make our lives a little bit easier, we're going to specify, at the beginning of every problem, a *universal set*. Any set defined later in the problem will have all its elements come from there). After that, there are closed figures inside which represent sets.
- To do things like we were doing in chapter 1 (where we want to talk about things like "all men are mortal"), we want to be able to represent different visual relationships between sets with symbols.
- One of those relationships is when one set is contained completely inside another. What does that mean in terms of elements?
- **Def:** Given two sets, $A$ and $B$, we say that $A$ is a *subset* of $B$ if every element of $A$ is also an element of $B$. This is denoted $A \subseteq B$.
- **Ex:** Consider the universe, $U$, which is the set of all buildings. Let $A = \{x \mid x \text{ is a museum}\}$. Let $B = \{x \mid x \text{ is a science museum}\}$. Then $B \subseteq A$. Note that $A \nsubseteq B$.
- **Def:** We'll say that $A$ is a proper subset of $B$ if $A$ is a subset of $B$, but is not equal to $B$. In this case, use the notation $A \subsetneq B$.
- What about if two sets overlap a little bit?
- **Def:** We'll call the area in between the sets the *intersection*. Formally, the intersection of sets $A$ and $B$ is $A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$. Note that the wedge explains the use of the $\cap$ for the intersection symbol.
- **Def:** If we have sets $X$ and $Y$ such that $X \cap Y = \emptyset$, then $X$ and $Y$ are said to be *mutually exclusive*.
- Another visual relationship that we want to encode is that of the combination of two sets.
- **Def:** Given two sets which overlap a little bit, we're going to call the set of stuff that's in either of the two sets the *union*.
- **Def:** Formally, given sets $A$ and $B$, the union is $A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$
- **Ex:** With $A = \{1, \pi, 10\}$ and $B = \{1, 2\}$, the union is $A \cup B = \{1, 2, \pi, 10\}$
- **Ex:** (This is Ex 1 in lecture guide) Let $U$ be the set of all real numbers. Consider the sets $R = \{1, \pi, 10\}$, $S = \{3, 9\}$, and $T = \{x \mid x \text{ is a whole number between } -3 \text{ and } 500\}$.
    1. What is $R \cup S$?
    2. What is $R \cap T$?
    3. What is $R \cap S$?
- How do we count things in a union?
- If we have disjoint sets (e.g. $X = \{a, b, c\}$ and $Y = \{d, e\}$), then the cardinality of $X \cup Y$ is $n(X \cup Y) = 5 = n(X) + n(Y)$.
- I claim this happens all the time. I.e. $n(X \cup Y) = n(X) + n(Y)$. Can anyone see why I'm wrong?
- Look at $X = \{1, 2, 3\}$ and $Y = \{3, 4, 5\}$. We see that $X \cup Y = \{1, 2, 3, 4, 5\}$, so $n(X \cup Y) = 5$, but $n(X) + n(Y) = 6$.
- What happened? Why is $n(A \cup B) \neq n(A) + n(B)$?

- We double counted the 3 when computing $n(A) + n(B)$. But in the union, the 3 only counts once.
- So what's the correct formula?
- The problem occurs when $A$ and $B$ have nontrivial intersection.
- In fact, the problem is with the intersection. When computing $n(A) + n(B)$, we count everything in $A \cap B$ twice.
- So how do we fix the problem?
- By making sure we count everything in $A \cap B$ only once! We do this by subtracting away everything in $A \cap B$
- We arrive at the following formula: $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.
- **Ex:** (This is Ex 2 in the lecture guide) With universe $U$ being the set of all whole numbers, consider the sets $A = \{1, -3, 4, -2, 0\}$ and $B = \{3, 2, 0, 1\}$, find
  1. $A \cap B$
  2. $n(A \cup B)$
- **Ex:** (This is Ex 3 in the lecture guide) Given $n(U) = 200$, $n(A) = 80$, and $n(B) = 70$, do the following:
  1. If $n(A \cap B) = 45$, find $n(A \cup B)$ and draw a Venn diagram illustrating the composition of the universal set $U$.
  2. If $n(A \cup B) = 150$, find $n(A \cap B)$ and draw a Venn diagram illustrating the composition of the universal set $U$.
- The last visual relationship (in a Venn diagram) we want to encode with notation is that of the "outside" of a set.
- **Def:** Given a universe, $U$, and set $X$, the *complement* of X is the set $X' = \{x \mid (x \in U) \wedge (x \notin X)\}$.
- **Ex:** Given the universe, $U = \{1, \ldots, 6\}$ and $X = \{2, 3, 5\}$, we have $X' = \{1, 4, 6\}$. Notice that $(X')' = X$. This is true no matter what we pick for $X$ and $U$.
- Note the relationship between the cardinalities of a set, $X$, it's complement, $X'$, and the universe, $U$ are related in the following way $n(U) = n(X) + n(X')$.
- This doesn't look earth-shattering, but it will help us a lot when we do probability. Often, the easiest way to count the number of ways something happens is to count the number of ways it doesn't happen.
- **Ex:** How many letters precede the letter $x$ in the English alphabet? 23.
- **Ex:** (This is Ex 4 in the lecture guide) Shade the following sets in the given Venn Diagram
  - Draw a Venn diagram with two sets partially overlapping. Have students shade $A \cap (B')$, $(A \cap B)'$, and $A' \cup B'$

<u>Section 2.2</u>: Applications of Venn Diagrams

- Venn Diagrams are one of many techniques used in an applied field of math called data visualization
- The goal is to better understand the data points you have.
- We won't be doing much data visualization, but we will be using Venn diagrams to solve counting problems similar to what we saw in 2.1
- Mostly, the type of problem we will see is called a cardinal number problem, where we are given some information about some sets and we use that information to count elements of other sets.

- To solve a cardinal problem, the book recommends the following steps (you don't have to do them, but they are kind of helpful):
  - Define a set for each property that the problem discusses
  - Draw a Venn diagram with as many overlapping circles as you have defined sets.
  - Write down the given cardinal numbers corresponding to the various sets
  - Starting with the innermost overlap, fill in each region of the Venn diagram with its cardinal number.
- Do examples 1 and 2 on the lecture guide
- Our next topic is that of De Morgan's Laws for sets
- We saw De Morgan's Laws previously for logical propositions $(\sim (p \wedge q)) \equiv ((\sim p) \vee (\sim q))$ and $(\sim (p \vee q)) \equiv ((\sim p) \wedge (\sim q))$.
- How do these logical propositions relate to sets?
- Given a universal set, $U$, and two subsets $A$ and $B$, let $p$ be the proposition "something is in $A$" and $q$ be the proposition "something is in $B$."
- Then De Morgan's laws applied to this particular case say that "something is not in both $A$ and $B$" is the same as saying "either something is not in $A$ or something is not in $B$"
- Rephrasing this in set language, we have that the complement of $A \cap B$ must be $A' \cup B'$.
- On the other hand, the other of De Morgan's laws states that "something is in neither $A$ nor $B$" is the same as "something isn't in $A$ and it's not in $B$"
- Rephrased in set language, we get that $(A \cup B)'$ is the same as $A' \cap B'$.
- We can see these equivalences via shading, as well (draw the sets and shade them on the board).
- We can see an example of this with Ex 3 in the lecture guide.
- Do the rest of the lecture guide.

<u>Section 2.3</u>: Introduction to Combinatorics

- It may surprise you to learn that our next topic is counting.
- Despite sounding innocuous, this is actually a fairly complicated branch of math.
- The book provides the example of you buying new clothes: you buy two new shirts, four new pairs of jeans, and three new pairs of shoes. How many possible new outfits have you created?
- One possible way of solving this problem is to set up a tree diagramming the possible outfits that you have.
- There are two possibilities for shirts, four possibilities for jeans, and three for shoes, leaving us with the following tree (which has 24 different outcomes).
- How did we end up with 24 outcomes? We multiplied our numbers of choices together.
- After picking a shirt and jeans, we had two groups of four (eight options total). After then picking shoes, we had eight groups of three (24 options total).
- This generalizes into looking at how many choices do you have if you have to make a bunch of consecutive choices together? Multiply the number of options you have for each choice.

- **Def:** This is called the *Fundamental Principle of Counting*: if you are making a series of $n$ choices and you have $c_1$ options for the first choice, $c_2$ options for the second choice,..., and $c_n$ options for your $n$th choice, you have $c_1 \cdot c_2 \cdots c_n$ options in total.
- **Ex:** (This is Ex 1 in lecture guide) A serial number for a phone has 5 letters followed by 2 numbers. How many phones can the manufacturer produce before they have to reuse a serial number? What if they want all of their serial numbers to never repeat a letter?
- **Ex:** (This is Ex 2 in the lecture guide) A homeowner interviews 6 people for a 3-bedroom house. How many different choices of pairings of people with bedrooms are there?
- **Ex:** How many possible 4-digit passcodes can you make for your phone? How many 6-digit? 10-digit? What if you want a 10-digit passcode to never repeat a letter?
- Notice with this last example, all we're doing is considering the number of ways to shuffle around all of the 0-9 digits.
- We're going to later see this called the number of permutations of 10 objects, but for now, we're going to focus on the number that we saw: $10 \cdot 9 \cdot 8 \cdots 1$
- **Def:** This operation where we start with a nonnegative whole number $n$ and multiply it by all of the smaller positive whole numbers is called the *factorial*. More formally, $n! = n \cdot (n-1) \cdot (n-2) \cdots 1$
- **Ex:** Compute $1!, 2!, 3!, 4!$. What about $0!$?

Section 2.4: Permutations and Combinations

- When given a counting problem, there are two fundamental questions that you need to ask before trying to answer the problem.
- First, is drawing done with or without replacement?
- Second, does order matter?
- Drawing with replacement means that if you select an item, it's still an option for being selected again.
- For instance, when we were creating passcodes for our phones, we noted that if you choose 4 for the first digit, you can choose 4 again. This is drawing with replacement.
- Drawing without replacement is the opposite. Once something has been selected, it can't be selected again.
- For instance, when we were creating an exec board of, say, 5 students from a club with 30 students, the drawing was done without replacement because once someone was selected to be president, they couldn't also be selected for treasurer.
- We say that order matters if drawing the same group of items in a different order is considered a different outcome.
- For instance, order mattered in our passcode: the passcodes 1234 and 4321 are different passcodes, even if they have the same group of items.
- We have yet to see anything where order doesn't matter. But you can come up with an example pretty easily. Say you're pulling books off of a bookshelf to loan to a friend. It doesn't matter if you draw *Harry Potter* before *The Da Vinci Code* or after. Your friend is getting both books just the same.
- Our two questions each have two answers, so we should know by now that there are four possible answer combinations.
- First, let's look at the case of drawing with replacement when order matters.

- This is our passcode example.
- If you are drawing $r$ items from a group of $n$ items with replacement and order matters, then we've already seen how to count the total number of possible outcomes.
- You have $n$ options for the first choice, $n$ items for the second choice, and so on, until you get to $n$ items for the $r$th choice.
- We know to multiply all of these together, so we get $n^r$ possible outcomes.
- Okay, that's not so bad. Now, let's look at the case when you are drawing without replacement when order matters.
- **Def:** The outcomes in this case are called *permutations*
- Let's look at an example first to see if we can figure out a general solution.
- **Ex:** At a local festival, 7 bands have auditioned for 4 slots. How many ways can the 7 bands be arranged among the four slots?
- Here, 7 is the size of the pool and 4 is the number of items that we're selecting. We started at 7 and multiplied it by the next three smaller whole numbers (so that we multiplied 4 numbers together total).
- If we only had 2 slots at our festival, we would have started at 7 and multiplied it by the next smaller number (so that we were multiplying 2 numbers together)
- In this way, we see that the size of the pool determines which number you start at.
- The number of items selected determines how many numbers get multiplied together.
- It would be nice to have a formula to count the number of permutations, however.
- Notice that what we were doing looked a lot like the factorial that we learned the other day. But instead of going all the way down to 1, we stopped after a certain point.
- Instead of thinking about it as stopping after a certain point, we want to think about it as starting with the full factorial, then canceling off some terms.
- In our 7 bands, 4 slots case, we wanted to cancel the last three terms of the factorial.
- In our 7 bands, 2 slots case, we wanted to cancel the last five terms of the factorial.
- If we had $n$ bands and $r$ slots, we would want to cancel the last $n-r$ terms of the factorial.
- This gives the general principle that if you are picking $r$ items from a pool of $n$ items without replacement and order matters, there are ${}_nP_r := \frac{n!}{(n-r)!}$ different permutations.
- **Ex:** (This is Ex 1 on lecture guide) At the Amazon Neighbors' Association Potluck, three emergency preparedness kits are raffled off. One has a flashlight set, another has a set of nonperishable food, the last is a case of bottled water. Supposing that all 5 attendees each have one ticket in the raffle (and drawing is done witout replacement), how many different ways are there for the prizes to be distributed?
- **Ex:** (This is Ex 2 on lecture guide) A golf team has 6 members. They tee off one at a time. In how many different orders can they tee off?
- Notice that we come across the quantity $0!$ here and we make good use of the fact that $0! = 1$. This tells us that despite the fact that $0! = 1$ is just a convention, it is an *good* convention. It's awfully useful and makes doing math easier. If we had chosen $0! = 0$, then we would have a lot of problems in defining and using factorials.

- Okay, next case. Let's examine how many outcomes we have when drawing is done without replacement and order doesn't matter.

- **Def:** The outcomes in this case are called *combinations*

- **Ex:** (This is Ex 3 in the lecture guide) Let's revisit the potluck raffle. Suppose that they are instead only raffling off two cases of water. Also, suppose that the five people in attendence are Alice, Bob, Charlie, Danielle, and Eve.

  1. Pretend, for the moment, that the two cases of water are different in some way (maybe water case 1 has 24 bottles and water case 2 has 48 bottles), so that order now matters (i.e. the outcome Alice, then Bob is different from the outcome Bob, then Alice). Explicitly write out each of the possible outcomes of the raffle using the table below:
  2. Now suppose that the water cases are indistinguishable. The list of possible outcomes that you came up with in the previous part now has some redundancy to it. Group together any outcomes from the previous part that produce the same outcome in this part. The groups represent the different combinations. How many items are in each group? How many groups do you have?
  3. What is the mathematical relationship between the number of permutations, the number of combinations, and the number of items in a group?
  4. Suppose again that the water cases are different in some way and now suppose that they are raffling off three cases, rather than two. Write down at least 10 of the possible outcomes (the permutations) in the table below.
  5. Finally, suppose that the three cases of water are again, indistinguishable. Write down a full group of permutations (from the previous part) that would result in the same outcome in this case. How large is your group? How many groups would you have, if you bothered to write all of them down?

- Students should do the first two parts on their own, I do the third part, students should do the last two parts.

- Breakdown: when counting the number of combinations (i.e. groups), we started with permutations, then divided by group size.

- How did we get group size? Well, it was the number of ways of a number of objects equal to the number of water cases could be permuted.

- But we know that's the factorial of the number of water cases.

- So we took our $n$ (total number of objects) and found the number of permutations when we selected $r$ (number of water cases) objects. Then, we divided that number by $r!$ to get the number of combinations.

- This gives the general principle that when you are selecting $r$ objects from a pool of $n$ objects without replacement and order doesn't matter, there are ${}_nC_r := \frac{n!}{(n-r)!r!}$ different combinations.

- **Ex:** (This is Ex 4 in the lecture guide) In how many different ways can you buy 3 CDs from a bin of 35 CDs?

- So now there's only one case left: drawing is done with replacement and order doesn't matter.

- This is hard enough that we're not going to worry about it in class. If you want to talk about it, come to office hours!

- The book gives a flowchart for decision-making regarding which counting techniques to use. You can use it, but I'm not sure it's very good.

  - When given a counting problem, first find all of the distinct categories from which things are being selected.

- Then, figure out how many possibilities there are in each category, using permutations, combinations, and with/without replacement as appropriate.

- **Ex:** (This is Ex 5 in the lecture guide) A deck of cards consists of 52 cards, each with one of four suits (spades, hearts, clubs, diamonds) and one of 13 denominations (2 - 10, jack, queen, king, ace). How many 5-card hands can you have with...
  - ...2 hearts and 3 spades?
  - ...2 hearts?
  - ...3 spades?
  - ...2 hearts or 3 spades?
  - ...2 kings and 2 queens and 1 card that's neither a king nor queen?

- **Ex:** (This is Ex 6 in the lecture guide) A race has 30 participants: 12 men and 18 women. The two podiums will consist of the top three runners in the men's and women's categories (in order). How many possible podium arrangements are there?

- Our last topic looks at permutations of items where some items are identical.

- **Ex:** (This is Ex 7 in the lecture guide) How many ways are there of arranging the letters in the word "saw"?

- **Ex:** (This is Ex 8 in the lecture guide) How many ways are there of arranging the letters in the word "see"?
  - First pretend the 'e's are distinct (say, $e_1$ and $e_2$).
  - Then permute everything.
  - Then notice that some of our permutations are identical. In fact, they're paired.
  - So we divide our total number of permutations in 2.

- **Ex:** (This is Ex 9 in the lecture guide) How many ways are there of arranging the letters in the word "epee"?
  - Notice that our identical arrangements now come in groups of $6 = 3!$.

- **Def:** This is our general principle: The number of *distinguishable permutations* of $n$ items in which $x$ items are identical, $y$ items are identical, $z$ items are identical, and so on, is $\frac{n!}{x!y!z!\cdots}$.

- **Ex:** (This is Ex 10 in the lecture guide) How many ways are there of arranging the letters in the word "Mississippi"?

- Worksheet time.

Section 2.5: Infinite Sets

- It might seem like a dumb question to ask: how can you tell if two sets have the same size?

- The obvious answer is that you count the elements! If they have the same number of elements, they're the same size. Otherwise, they're not.

- This has its limitations, however. Are the set of whole numbers and the set of real numbers the same size? They both have infinitely many elements after all.

- We're going to see that there are different ways of measuring infinity, however.

- Here's a better way of deciding if two sets have the same size.

- **Def:** Two sets, $A$ and $B$, are said to be in *one-to-one correspondance* (or *equivalent*) if it's possible to pair every element of $A$ with exactly one element of $B$ without leaving out any elements of either set.

- **Ex:** Consider the sets $A = \{2, 3, \pi\}$, $B = \{\text{John Williams, JS Bach, Count Basie}\}$, and $C = \{\alpha, \beta, \gamma, \delta\}$. Which of these sets are equivalent?
- **Ex:** Consider the sets $X = \{1, 2, \ldots, 50\}$ and $Y = \{1, 3, 5, \ldots, 99\}$. Are $X$ and $Y$ equivalent?
  - Make sure to use the relationship $n \leftrightarrow (2n - 1)$
- **Ex:** Let $X$ be a finite set. Is it possible for a proper subset of $X$ to be equivalent to $X$?
- No! Proper subsets have less stuff, so they can't be put into one-to-one correspondance.
- Now consider the natural numbers: i.e. all of the nonnegative whole numbers. $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. Also consider the even numbers $E = \{2, 4, 6, 8, \ldots\}$. Do $\mathbb{N}$ and $E$ have a one-to-one correspondance?
- This is pretty weird. We have a set which is the same size as one of its proper subsets.
- **Def:** We say that $\mathbb{N}$ is *countably infinite*, or *countable*.
- What other sets are countable?
- Let's look at $\mathbb{Z}$.
- Let's look at $\mathbb{Q}$.
- Let's look at the $\mathbb{R}$, but only the real numbers between 0 and 1.
- What about the intervals $[0, 1]$ and $[0, 2]$?